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AN APPLICATION IN LINEAR CODES RELATING NUMERICAL SEQUENCES AND THE CASORATIAN MATRIX

Lara Nicoletti Sotoma
Federal University of Mato Grosso do Sul, Insitute of Mathematics, Campo Grande, Mato Grosso do Sul, Brasil
laransotoma@gmail.com
Marcos Vinicius Pereira Spreafico
Federal University of Mato Grosso do Sul, Insitute of Mathematics, Campo Grande, Mato Grosso do Sul, Brasil
marcos.spreafico@ufms.br
Elen Viviani Pereira Spreafico
Federal University of Mato Grosso do Sul, Insitute of Mathematics, Campo Grande, Mato Grosso do Sul, Brasil
elen.spreafico@ufms.br

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Resumo

Neste artigo, a relação entre a representação matricial e uma sequência de recorrência linear homogênea de ordem 2 é explorada, utilizando a abordagem de sistemas fundamentais. Além disso, utilizando a relação apresentada, uma aplicação em teoria de codificação é introduzida considerando a representação matricial como uma matriz geradora de um código linear, e o desempenho do código é analisado sob a perspectiva da correção de erros.

Palavras-chave: Sequências Recorrentes, Sistema Fundamental, matriz Casoratiana, Códigos Lineares.

Abstract

In this paper, the relationship between the matrix representation and a homogeneous linear recurrence sequence of order 2 is explored using the fundamental systems approach. Furthermore, using the presented relationship, an application in coding theory is introduced, considering the matrix representation as a generating matrix of a linear code, and the performance of the code is analyzed from the perspective of error correction.

Keywords: Recurrence Sequences, Fundamental System, Casoratian matrix, Linear Codes.

1 Introduction

Throughout history, the human race has sought to describe the world in terms of shapes and patterns. In this context, and to understand it, it is necessary to observe a phenomenon and try to describe it based on some existing pattern.

In terms of numerical sequence, the well-known Fibonacci sequence is the first discrete population model and, also, the limit of the quotient of its two consecutive numbers (golden ratio) describes the shape of plans and architectural constructions. Other sequences related to the Fibonacci sequence, such as the Lucas, Fibonacci-Lucas, and Pell numbers, offer additional compelling applications across various domains including computer science, art, and finance. Therefore, a recursive numerical sequence can be applied as a didactic object, [4, 16].

On the other side, a recurrence numerical sequence can be seen as a difference equation. When we make this association, we have a linear algebra structure to study the properties of these numerical sequences, [8, 10, 1, 11]. The main algebraic tool for the study of sequences and their solutions is the Casoratian matrix (see more in [20] and the references therein). This matrix has multiplicative properties that make it possible to derive properties of the numerical sequence associated with it.

The concept of a Casoratian matrix was introduced by Felice Casorati and can be described as a discrete version of the Wroskian matrix, used in the study of different types of differential equations.

This article explores the relationship between the matrix representation of a linear homogeneous recurrence sequence of order 2. More precisely, we will build an application on linear codes using the relationship with the Casoratian matrix and the linear homogeneous recurrence sequence of order 2. Given a homogeneous recurrence sequence of order 2, our approach consists in associating this sequence with their Casoratian matrix, by using the properties of the fundamental system (see, for instance, [24, 23]).

Moreover, a specific linear homogeneous recurrence sequence of order 2 is provided and a matrix code derived from it is built. Error-correcting codes play a fundamental role in ensuring the reliability of data transmission and storage by detecting and correcting errors that occur in noisy communication channels, which is crucial in modern digital systems. Linear codes, which are defined through generator or parity-check matrices, are essential in coding theory because they provide algebraic structures that simplify both the encoding and decoding processes, enabling efficient error detection and correction [2, 3, 18, 19, 26, 27].

The basic concepts of Coding Theory was introduced by Shannon in [22]. However, the approach of linear codes was developed by Hamming in [9]. In linear codes, each message is represented as a vector and, multiplied by a generator matrix (denoted by$G$), to produce a coded word. This structure allows redundancy to be added systematically, making it possible not only to detect but also to correct errors during transmission. Complementing this structure, the parity-check matrix (denoted by H) is used upon receiving the message to verify the consistency of the data and to locate possible errors. These concepts form the basis of classical codes such as Hamming codes and cyclic codes, both widely used in communication systems and digital data storage.

New approaches have explored the properties of recurrence sequences and their matrix representations. In [17] the authors introduced a generalization of the Fibonacci matrix of order 2, given by a $n \times n$ k-order Fibonacci matrix associated with the recurrence relation $F_n = F_{n-1} + \cdots + F_{n-r}$, with initial conditions $F_0 = F_1 = \cdots = F_{r-2}$ and $F_{r-1} = 1$. Similar to the Casoratian matrix associated to Fibonacci sequence, using the powers of the k-order Fibonacci matrix, the article presents explicit formulas for the elements of the sequence, as well as combinatorial identities.

Regarding the relationship between numerical sequences and matrices, we have Pascal's matrix (see [15] and references therein). The generalization of Pascal's matrix, called k-augmented Pascal matrix, was given in [14, 13]. The authors establishes links with important combinatorial numbers (Fibonacci,Stirling and Eulerian numbers), and provides a decomposition that helps to understand its structure. Other article that involves sequences and matrices is [12], where the authors presents a matrix approach to deriving identities for sequences of Sheffer polynomials.

In [26], the author introduced codes based on Fibonacci Q-matrix, [25]. This matrix is associated with the well-known Fibonacci sequence, but its generalization allows the construction of codes with special properties of symmetry and robustness.

The central idea is to use the algebraic structure of the Fibonacci matrix to generate new coding schemes, in which the original message is transformed through multiplication by powers of the Casoratian matrix. These codes exhibit particularly interesting mathematical behavior, such as connections with classical formulas (e.g., the generalization of Cassini's identity) and the possibility of extending the methods to Lucas sequences and other types of recurrence (see more in [2, 3]). A point that was not addressed by the authors is that the matrix used in the process of coding and decoding is simply the Casoratian matrix. Therefore, the extension can be done by using the matrix properties arising from the Theory of Difference Equations. This point of view is explored in this article.

The article is organized as follows. The first section provides the relationship among the Casoratian matrix, the fundamental system, and a homogeneous linear recurrence sequence of order 2. Section 2 presents a new construction of a matrix code using the Casoratian matrix. Moreover, the coding and decoding method introduced by [2] is generalized to the homogeneous linear recurrence sequence of order 2. Furthermore, the error correction capability of these codes using the limit of the quotient of two consecutive terms of the sequence associated with it is provided. Finally, conclusions are stated.

2 Casoratian matrix and a Homogeneous Linear Recurrence Sequences of order 2

Let $\{v_n\}_{n\geq 0}$ be a sequence defined by a linear recurrence relation of order 2

$$v_n = a_0 v_{n-1} + a_1 v_{n-2}, \tag{2.1}$$

for $n \geq 2$, $a_1 \neq 0$, and initial values $v_0 = \alpha_0$ and $v_1 = \alpha_1$. Consider Equation (2.1) as a difference equation. A solution of this difference equation is a sequence that satisfy Equation (2.1). According to [8], the set of solutions of Equation (2.1) is a vector space over real or complex set of numbers. Suppose that $\{v_n^{(1)}\}_{n\geq 0}$, $\{v_n^{(2)}\}_{n\geq 0}$ are 2 solutions of Equation (2.1). We say that $\{v_n^{(1)}\}_{n\geq 0}$, $\{v_n^{(2)}\}_{n\geq 0}$ are *linearly dependents* if there exists 2 scalars β_1, β_2, with $(\beta_1, \beta_2) \neq (0, 0)$, such that

$$\beta_1 v_n^{(1)} + \beta_2 v_n^{(2)} = 0, \text{ for every } n \geq 0,$$

and $\{v_n^{(1)}\}_{n\geq 0}$, $\{v_n^{(2)}\}_{n\geq 0}$ are said *linearly independents* if they are not linearly dependent. If the sequences $\{v_n^{(1)}\}_{n\geq 0}$, $\{v_n^{(2)}\}_{n\geq 0}$ are linearly independent, we say that the set $\{\{v_n^{(1)}\}_{n\geq 0}, \{v_n^{(2)}\}_{n\geq 0}\}$ is a fundamental system of solutions of Equation (2.1).

Consider the sequence $\{w^{(1)}\}_{n\geq 0}$ defined by

$$w_n^{(1)} = a_0 w_{n-1}^{(1)} + a_1 w_{n-2}^{(1)}, \tag{2.2}$$

for $n \geq 2$, $a_1 \neq 0$, and initial values $w_0 = 1$ and $w_1 = 0$. Consider the sequence $\{w^{(2)}\}_{n\geq 0}$ defined by

$$w_n^{(2)} = a_0 w_{n-1}^{(2)} + a_1 w_{n-2}^{(2)}, \tag{2.3}$$

for $n \geq 2$, $a_1 \neq 0$, and initial values $w_0 = 0$ and $w_1 = 1$. These sequences are solution of Equation (2.1).

Next, we will provide by using a matrix concept that the set $\{\{w_n^{(1)}\}_{n\geq 0}, \{w_n^{(2)}\}_{n\geq 0}\}$ is a fundamental system of solutions of Equation (2.1), (see more in [24, 6, 23]).

The Casoratian matrix associated to $\{\{w_n^{(1)}\}_{n\geq 0}, \{w_n^{(2)}\}_{n\geq 0}\}$ is defined by

$$\hat{C}(k) = \begin{bmatrix} w_k^{(1)} & w_k^{(2)} \\ w_{k+1}^{(1)} & w_{k+1}^{(2)} \end{bmatrix}. \tag{2.4}$$

The determinante of the Casoratian matrix is called the Casoratian. The Casoratian of $C(k)$ is given by

$$C(k) = \begin{vmatrix} w_k^{(1)} & w_k^{(2)} \\ w_{k+1}^{(1)} & w_{k+1}^{(2)} \end{vmatrix} = w_k^{(1)} w_{k+1}^{(2)} - w_k^{(2)} w_{k+1}^{(1)}. \tag{2.5}$$

There are important properties of the Casoratian matrix. For example, the Casoratian matrix verifies, for every n and m, $\hat{C}(m+n) = \hat{C}(m)\hat{C}(n)$, (see more in [24, 6, 23]). Moreover, in this article we will use the fact of the Casoratian matrix is similar a companion matrix (see more in [5]). Consider the sequence $\{v_n\}_{n\geq 0}$ solution of Equation (2.1). Observe that we can rewrite this recurrence relation in matrix form:

$$\begin{bmatrix} v_n \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & a_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_{n-1} \\ v_{n-2} \end{bmatrix}.$$

The matrix $A = \begin{bmatrix} a_0 & a_1 \\ 1 & 0 \end{bmatrix}$ is called the companion matrix associated to $\{v_n\}_{n\geq 0}$. By using the recursive method we obtain the vector $\begin{bmatrix} v_n \\ v_{n-1} \end{bmatrix}$ in terms of the power of A, as

$$\begin{bmatrix} v_n \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & a_1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} \alpha_1 \\ \alpha_0 \end{bmatrix}.$$

We can establish, by induction on n, the following relation $A^n = J\hat{C}(k)J$, where $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Therefore, since

$$C(k) = (det(A))^n = (-1)^n a_1 \neq 0,$$

then we prove that $\{w_n^{(1)}\}_{n\geq 0}$, $\{w_n^{(2)}\}_{n\geq 0}$ are linearly independent and the set $\{\{w_n^{(1)}\}_{n\geq 0}$, $\{w_n^{(2)}\}_{n\geq 0}\}$ is a fundamental system of solutions of Equations (2.1).

Moreover, we can also prove that

$$w_n^{(1)} = a_1 w_{n-1}^{(2)}, \tag{2.6}$$

for each $n \geq 1$. (see, for instance, [24, 6]), and then

$$A^n = \begin{bmatrix} a_0 & a_1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_k^{(1)} & w_k^{(2)} \\ w_{k+1}^{(1)} & w_{k+1}^{(2)} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 w_{k-1}^{(2)} & w_k^{(2)} \\ a_1 w_k^{(2)} & w_{k+1}^{(2)} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that if $a_1 = 1$, then the Casoratian matrix is equal to the matrix

$$\begin{bmatrix} w_k^{(1)} & w_k^{(2)} \\ w_{k+1}^{(1)} & w_{k+1}^{(2)} \end{bmatrix} = \begin{bmatrix} w_{k-1}^{(2)} & w_k^{(2)} \\ w_k^{(2)} & w_{k+1}^{(2)} \end{bmatrix}.$$

Denote

$$Q_k = \begin{bmatrix} w_{k-1}^{(2)} & w_k^{(2)} \\ w_k^{(2)} & w_{k+1}^{(2)} \end{bmatrix}.$$

For example, consider the Fibonacci sequence given by

$$F_n = F_{n-1} + F_{n-2}, \tag{2.7}$$

for $n \geq 2$, and initial values $F_0 = 0$ and $F_1 = 1$. The fundamental system is given by the sequences $\{F^{(1)}\}_{n\geq 0}$ and $\{F^{(2)}\}_{n\geq 0}$ defined by $F_n^{(1)} = F_{n-1}^{(1)} + F_{n-2}^{(1)}$, for $n \geq 2$, $a_1 \neq 0$, and initial values $F_0^{(1)} = 1$ and $F_1^{(1)} = 0$, and $F_n^{(2)} = F_{n-1}^{(2)} + F_{n-2}^{(2)}$, for $n \geq 2$, $a_1 \neq 0$, and initial values $F_0^{(2)} = 0$ and $F_1^{(1)} = 1$. Observe that, in this case, $F_n^{(2)} = F_n$, and the Casoratian matrix is equal to the matrix

$$\begin{bmatrix} F_{k-1}^{(2)} & F_k^{(2)} \\ F_k^{(2)} & F_{k+1}^{(2)} \end{bmatrix} = \begin{bmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{bmatrix}.$$

Moreover,

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

In addition, we can derive the Cassini identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ by using the determinant.

Now, consider the Pell sequence given by

$$P_n = 2P_{n-1} + P_{n-2}, \tag{2.8}$$

for $n \geq 2$, and initial values $P_0 = 0$ and $P_1 = 1$. The fundamental system is given by the sequences $\{P^{(1)}\}_{n\geq 0}$ and $\{P^{(2)}\}_{n\geq 0}$ defined by $P_n^{(1)} = P_{n-1}^{(1)} + P_{n-2}^{(1)}$, for $n \geq 2$, $a_1 \neq 0$, and initial values $P_0^{(1)} = 1$ and $P_1^{(1)} = 0$, and $P_n^{(2)} = P_{n-1}^{(2)} + P_{n-2}^{(2)}$, for $n \geq 2$, $a_1 \neq 0$, and initial values $P_0^{(2)} = 0$ and $P_1^{(1)} = 1$. Observe that, in this case, $P_n^{(2)} = P_n$, and the Casoratian matrix is equal to the matrix

$$\begin{bmatrix} P_{k-1}^{(2)} & P_k^{(2)} \\ P_k^{(2)} & P_{k+1}^{(2)} \end{bmatrix} = \begin{bmatrix} P_{k-1} & P_k \\ P_k & P_{k+1} \end{bmatrix}.$$

Similarly, we have

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{k-1} & P_k \\ P_k & P_{k+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} P_{k+1} & P_k \\ P_k & P_{k-1} \end{bmatrix}.$$

Moreover, we can derive a Cassini identity type for the Pell numbers, $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$, by using the determinant.

3 An application: matrix codes using sequences

This section presents an application on linear codes using the generalization of the Fibonacci sequence of order 2. Let $\{v_n\}_{n\geq 0}$ be a sequence defined by a linear recurrence relation of order 2

$$v_n = a_0 v_{n-1} + v_{n-2}, \tag{3.1}$$

for $n \geq 2$, where a_0 is an arbitrary positive integer and initial values $v_0 = 0$ and $v_1 = 1$. Therefore, the fundamental system is given by the sequences $\{v^{(1)}\}_{n\geq 0}$ and $\{v^{(2)}\}_{n\geq 0}$ defined by $v_n^{(1)} = a_0 v_{n-1}^{(1)} + v_{n-2}^{(1)}$, for $n \geq 2$, $a_1 \neq 0$, and initial values $v_0^{(1)} = 1$ and $v_1^{(1)} = 0$, and $v_n^{(2)} = a_0 v_{n-1}^{(2)} + v_{n-2}^{(2)}$, for $n \geq 2$, and initial values $v_0^{(2)} = 0$ and $v_1^{(2)} = 1$. Thus, for $n \geq 1$,

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_{n-1} & v_n \\ v_n & v_{n+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} v_{n+1} & v_n \\ v_n & v_{n-1} \end{bmatrix} = Q_n.$$

Define the sequence indexed by negative integers $\{v_n\}_{n<0}$ as $v_{-n} = (-1)^{n+1}v_n, n \geq 1$, to reach our goal. Therefore, we obtain

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}^{-n} = \begin{bmatrix} v_{-n-1} & v_{-n} \\ v_{-n} & v_{-n+1} \end{bmatrix}.$$

Then, if $n > 0$ is even,

$$Q_{-n} = \begin{bmatrix} v_{n-1} & -v_n \\ -v_n & v_{n+1} \end{bmatrix}.$$

Otherwise, if $n > 0$ is odd,

$$Q_{-n} = \begin{bmatrix} -v_{n-1} & v_n \\ v_n & -v_{n+1} \end{bmatrix}.$$

The matrices Q_n and Q_{-n} play an important rule on coding and decoding process. Next, we present the construction of de matrix codes where the code words are obtained by multiplying the Casoratian matrix Q_n on the right. The decoding process depends on the Casoratian matrix Q_{-n}. This coding and decoding process generalizes the algorithm introduced in [2] for all sequences defined by Equation (3.1), which includes Fibonacci, Lucas, and Pell numbers.

3.1 Matrix Code

Consider a message given by a matrix B of order 2×2, with integer positive entries:

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

The coding process is described by multiplication of the matrix B and Q_n, obtaining the code matrix $C = BQ_n$. Then

$$C = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} v_{n+1} & v_n \\ v_n & v_{n-1} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}.$$

The decoding process is given by $B = CQ_{-n}$, or

$$B = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} Q_{-n}.$$

For odd n, we obtain

$$B = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} v_{n-1} & -v_n \\ -v_n & v_{n+1} \end{pmatrix},$$

or equivalently,

$$\begin{cases} b_1 = c_1 \cdot v_{n-1} - c_2 \cdot v_n, \\ b_2 = -c_1 \cdot v_n + c_2 \cdot v_{n+1}, \\ b_3 = c_3 \cdot v_{n-1} - c_4 \cdot v_n, \\ b_4 = -c_3 \cdot v_n + c_4 \cdot v_{n+1}. \end{cases}$$

Since the entries of matrix B are positive integers,

$$\begin{cases} b_1 = c_1 \cdot v_{n-1} - c_2 \cdot v_n > 0, \\ b_2 = -c_1 \cdot v_n + c_2 \cdot v_{n+1} > 0, \\ b_3 = c_3 \cdot v_{n-1} - c_4 \cdot v_n > 0, \\ b_4 = -c_3 \cdot v_n + c_4 \cdot v_{n+1} > 0. \end{cases}$$

Then,

$$\begin{aligned} c_1 \cdot v_{n-1} - c_2 \cdot v_n &> 0 \\ c_1 \cdot v_{n-1} &> c_2 \cdot v_n \\ c_1 &> c_2 \cdot \frac{v_n}{v_{n-1}}, \end{aligned}$$

$$\begin{aligned} -c_1 \cdot v_n + c_2 \cdot v_{n+1} &> 0 \\ c_2 \cdot v_{n+1} &> c_1 \cdot v_n \\ c_2 \cdot \frac{v_{n+1}}{v_n} &> c_1. \end{aligned}$$

Therefore, we obtain

$$\begin{gathered} c_2 \cdot \frac{v_n}{v_{n-1}} < c_1 < c_2 \cdot \frac{v_{n+1}}{v_n} \\ \frac{v_n}{v_{n-1}} < \frac{c_1}{c_2} < \frac{v_{n+1}}{v_n}. \end{gathered}$$

Similarly,

$$\frac{v_n}{v_{n-1}} < \frac{c_3}{c_4} < \frac{v_{n+1}}{v_n}.$$

The ratio between two successive terms of a sequence $\{a_n\}_{n\geq 0}$ is defined by $q_n = \frac{a_{n+1}}{a_n}$. Since $lim_{n\to\infty} \frac{v_n}{v_{n-1}} = \psi = \frac{a_0+\sqrt{a_0^2+4}}{2}$, (see, for instance, Theorem 7 in [7]), follows that $c_1 \approx \psi c_2$ e $c_3 \approx \psi c_4$. Similarly, for even n, we obtain $c_1 \approx \psi c_2$ e $c_3 \approx \psi c_4$.

Observe that the correction of errors depends of the limit of the ratio between two successive terms of a sequence defined by Equation (3.1).

3.2 Error Correction

Now, we show how to detect and correct errors in matrix codes. The detection will be performed using the determinant. Thus, in the case where there are no errors, we should have $det(C) = det(A^n) \cdot det(B)$, which gives $det(C) = (-1)^n \cdot det(B)$.

Now, consider the possibility of an error in one of the entries of the code matrix C. There are four possible cases:

$$\begin{pmatrix} x & c_2 \\ c_3 & c_4 \end{pmatrix}, \quad \begin{pmatrix} c_1 & y \\ c_3 & c_4 \end{pmatrix}, \quad \begin{pmatrix} c_1 & c_2 \\ z & c_4 \end{pmatrix}, \quad \begin{pmatrix} c_1 & c_2 \\ c_3 & w \end{pmatrix}$$

where x, y, z, w represent the errors. In each case, we have

$$\begin{cases} xc_4 - c_2c_3 = (-1)^n \cdot det(B), \\ c_1c_4 - yc_3 = (-1)^n \cdot det(B), \\ c_1c_4 - c_2z = (-1)^n \cdot det(B), \\ c_1w - c_2c_3 = (-1)^n \cdot det(B). \end{cases}$$

Then x, y, z, w can be determined by

$$\begin{cases} x = \dfrac{(-1)^n \cdot det(B) + c_2c_3}{c_4}, \\ y = \dfrac{-(-1)^n \cdot det(B) + c_1c_4}{c_3}, \\ z = \dfrac{-(-1)^n \cdot det(B) + c_1c_4}{c_2,}, \\ w = \dfrac{(-1)^n \cdot det(B) + c_2c_3}{c_1}. \end{cases}$$

Now, there are six possibilities when the code matrix presents errors in two of its entries:

$$\begin{pmatrix} x & y \\ c_3 & c_4 \end{pmatrix}, \begin{pmatrix} c_1 & c_2 \\ z & w \end{pmatrix}, \begin{pmatrix} x & c_2 \\ z & c_4 \end{pmatrix}, \begin{pmatrix} c_1 & y \\ c_3 & w \end{pmatrix}, \begin{pmatrix} x & c_2 \\ c_3 & w \end{pmatrix}, \begin{pmatrix} c_1 & y \\ z & c_4 \end{pmatrix},$$

where x, y, z, w represent the errors.

These six possibilities can be divided into two cases:

i. Errors in the same row:

$$\begin{pmatrix} x & y \\ c_3 & c_4 \end{pmatrix}, \begin{pmatrix} c_1 & c_2 \\ z & w \end{pmatrix}.$$

In these two cases, when calculating the determinant, we obtain

$$\begin{cases} xc_4 - yc_3 = (-1)^n \cdot det_B \\ c_1 w - c_2 z = (-1)^n \cdot det_B \end{cases}.$$

These are equations with infinitely many solutions. To determine the elements x, y, z, w, we use the relations $c_1 \approx \psi c_2$ and $c_3 \approx \psi c_4$, which give us $x \approx \psi y$ and $z \approx \psi w$. We also consider that the elements of the code matrix are integers.

ii. Errors in different rows:

$$\begin{pmatrix} x & c_2 \\ z & c_4 \end{pmatrix}, \begin{pmatrix} c_1 & y \\ c_3 & w \end{pmatrix}, \begin{pmatrix} x & c_2 \\ c_3 & w \end{pmatrix}, \begin{pmatrix} c_1 & y \\ z & c_4 \end{pmatrix}.$$

In all these cases, we simply consider that $c_1 \approx \psi c_2$ and $c_3 \approx \psi c_4$, and that the elements c_1, c_2, c_3, c_4 are integers.

Now, considering the case where the code matrix presents three errors, there are four possibilities.

$$\begin{pmatrix} c_1 & y \\ z & w \end{pmatrix}, \begin{pmatrix} x & c_2 \\ z & w \end{pmatrix}, \begin{pmatrix} x & y \\ c_3 & w \end{pmatrix}, \begin{pmatrix} x & y \\ z & c_4 \end{pmatrix}.$$

In all these possibilities, we use the relation $c_1 \approx \psi c_2$, or the relation $c_3 \approx \psi c_4$, along with the fact that the elements of the code matrix are integers to determine the element that is in the same row as the error-free element. Then, the matrix reduces to a case from the previous group (i).

Finally, the case where the matrix has errors in all entries is uncorrectable. Thus, out of the 15 possible errors - 4 single errors, 6 double errors, 4 triple errors, and 1 total error - 14 cases can be corrected.

Example 3.1. Consider a mensage gave in a matrix B with integers entries, coding by the Fibonacci code of $n = 12$. The determinant of B is $det_B = 26$ and the received matrix C is given by

$$C = \begin{pmatrix} 3304 & 2024 \\ 23183 & 14328 \end{pmatrix}.$$

Now, we can verify if there are errors in matrix C. Observe that

$$det_C = 3304 \cdot 14328 - 23183 \cdot 2024$$
$$det_C = 47339712 - 46922392 = 417320.$$

Since $det_C \neq (-1)^{12} \cdot det_B$ there is an error in the code matrix. For $n = 12$, the quotient $\frac{f_{13}}{f_{12}} = 1.61805$ is close to 4 decimal digits of ϕ. Then $\frac{3304}{2024} = 1.6324...$, which is not very close to ϕ, implies that the error is in the first line. Replacing c_1 with x in $det_C = (-1)^n \cdot det_B$, we obtain $x = 3274.8756...$, which is not an integer. Therefore, the error is in c_2. Replacing c_2 with y, we have

$$det_C = 3304 \cdot 14328 - 23183 \cdot y$$
$$y = \frac{47339686}{23183}$$
$$y = 2042.$$

Then, since y is a integer number, the entry c_2 must be corrected by y.

4 Concluding remarks and perspective

This work presented the relationship among the Casoratian matrix, the fundamental system and a homogeneous linear recurrence sequence of order 2. As an application, we construct a matrix code using the Casoratian matrix, showing the coding and decoding method. Furthermore, we studied the correction of errors using the limit of the quotient of two consecutive terms of the sequence associated with it.

It seems to us that this work is a preliminary article to understand the deeper algebraic relationship between numerical sequences and matrices. In addition, since each result presented holds for homogeneous linear recurrence sequence of order r, $r \geq 2$, this article contributes with the extensions of matrix code using other numerical sequences.

In fact, there are two main principles to build a matrix-based code. Given an invertible matrix A, we need to know the n-th power elements of A and the n-th power elements of A^{-1}. Detecting the error depends on the knowledge of the determinant of the determinant of A^{-n}. Moreover, here, the error correction was done by one-to-one comparison, since we know the relationship between the entries of matrix A. Therefore, this reasoning can be applied to other matrices.

For example, since the generalized Fibonacci matrix introduced by [17] is a superior triangular matrix which the non zero entries are the Fibonacci numbers, the construction of matrix-based code is the same presented here. Hence, since the correction of the error depends on the knowledge of the relationship between the entries, it is possible correct at least one error.

This article does not discuss the computational terms of this code, but the process involves algorithms with computational cost (to find inverse matrix and determinants).

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Data Availability

Not applicable.

Author Contributions

All authors have contributed equally to the development and writing of this article.

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