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A DIFFERENT PROOF OF A PROPOSITION ABOUT WEAKLY MODULAR FUNCTIONS

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Abstract

We present a new proof of a well-known result concerning weakly modular functions, avoiding lemmas about the behavior of specific elements. The approach relies on the finite induction technique, offering an alternative interpretation that simplifies the proof and enriches analytic number theory.

Keywords: Weakly modular functions, finite induction technique.

1 Introduction

Analytic number theory is a branch of mathematics that investigates arithmetic properties using methods from complex analysis, playing a central role in the understanding of deep structures within the integers. In this context, weakly modular functions arise as particularly interesting objects, as they extend the classical notion of modular functions by allowing poles at points within the fundamental domain. These functions have significant applications in various areas, including the theory of modular forms, integer partitions, and theoretical physics. In this article, we present an original approach to proving a well-known result concerning weakly modular functions. The proof is usually omitted or lacks details, and the approach that will be presented here uses simpler reasoning and a well-known technique, finite induction. This perspective not only simplifies certain technical aspects but also provides a new structural interpretation of the behavior of these functions, contributing to a deeper understanding of the theory.

2 The modular group

The modular group is an object that emerges naturally in various areas, including geometry, number theory, and even theoretical physics. At its core, it is a group that acts on the complex plane in a structured and elegant way, often represented by their effects on shapes like circles or grids. The transformation preserves specific properties, such as angles and distances, making the modular group a tool for studying symmetry. In geometry, the modular group helps to describe how surfaces like a torus can be transformed. Even in physics, particularly in string theory and quantum field theory, its symmetries offer insights into the structure of space-time.

2.1 Definitions

Definition 2.1. Let H denote the upper half plane of $\mathbb{C}$, that is,

$$H = \{z \in \mathbb{C} \mid Im(z) > 0\}.$$

Definition 2.2. Let $SL_2(\mathbb{R})$ be the group of real 2×2 matrices $A \in \mathscr{M}_2(\mathbb{R})$ such that $\det(A) = 1$, i.e.,

$$SL_2(\mathbb{R}) = \{A \in \mathscr{M}_2(\mathbb{R}) \mid \det(A) = 1\}.$$

Definition 2.3. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ and $z \in \tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We define the action of $SL_2(\mathbb{R})$ on $\tilde{\mathbb{C}}$ by

$$\begin{aligned} \alpha : \; SL_2(\mathbb{R}) \times \tilde{\mathbb{C}} &\to \tilde{\mathbb{C}} \\ (g, z) &\mapsto \alpha(g, z) = \frac{az + b}{cz + d}. \end{aligned}$$

However, in order to clean the notations, we will denote the above action by

$$\alpha(g, z) = gz. \tag{2.1}$$

Note that, if $z = x + iy$, then

$$gz = \frac{az + b}{cz + d} \cdot \frac{cz - d}{cz - d} = \frac{(ax + b + iay)(cx + d - icy)}{|cz + d|^2}.$$

Hence, we obtain

$$\operatorname{Im}(gz) = \frac{(ad - bc)\operatorname{Im}(z)}{|cz + d|^2} = \frac{Im(z)}{|cz + d|^2}. \tag{2.2}$$

This shows that H is stable under the action of $SL_2(\mathbb{R})$, since $\operatorname{Im}(z) > 0$ implies $\operatorname{Im}(gz) > 0$. In other words, the upper half-plane is mapped onto itself under the action.

Also, note that

$$-gz = \frac{-az-b}{-cz-d} = \frac{-(az+b)}{-(cz+d)} = \frac{az+b}{cz+d} = gz, \tag{2.3}$$

for all $g \in SL_2(\mathbb{R})$.

Definition 2.4. We define $1 = \mathrm{Id} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ and $-1 = -\mathrm{Id} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in SL_2(\mathbb{R})$

Definition 2.5. We define the quotient group $PSL_2(\mathbb{R})$ by

$$PSL_2(\mathbb{R}) = \frac{SL_2(\mathbb{R})}{\{\pm 1\}}.$$

It is called the Projective Special Linear Group.

Definition 2.6. Let $SL_2(\mathbb{Z})$ be the subgroup of $SL_2(\mathbb{R})$ consisting of the matrices with coefficients in $\mathbb{Z}$. It is a discrete subgroup of $SL_2(\mathbb{R})$.

Definition 2.7. The group

$$G = PSL_2(\mathbb{Z}) = \frac{SL_2(\mathbb{Z})}{\{\pm 1\}}$$

is called the modular group. It is the image of $SL_2(\mathbb{Z})$ in $PSL_2(\mathbb{R})$.

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. We often use the same symbol to denote its image in the modular group G.

2.2 Fundamental domain of the modular group

Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in G$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in G$. Note that

$$\begin{gathered} Sz = -1/z, \\ Tz = z+1, \\ S^2 = -1 = 1 \in G, \\ (ST)^3 = -1 = 1 \in G. \end{gathered}$$

Definition 2.8. Let two points $z_1, z_2 \in H$. We say they are congruent modulo G if there exists $g \in G$ such that $z_1 g = z_2$.

Definition 2.9. Let G be a group and X be a topological space such that G acts on X. Let $F \subseteq X$. We say that F is a fundamental domain for such an action if it satisfies the following conditions:

1. Any $x \in X$ is congruent modulo G to some point in F;
2. No two interior points in F are congruent modulo G.

Definition 2.10. Let $z \in H$. We define the *stabilizer of z in G* as the set

$$G_z = \{g \in G \mid gz = z\},$$

i.e., the set of elements of G which leave z unchanged under the action.

Definition 2.11. Let D be the subset of H formed of all points $z \in H$ such that $|z| \geq 1$ and $|\operatorname{Re}(z)| \leq 1/2$, i.e.,

$$D = \{z \in H;\ |z| \geq 1 \ \wedge \ |\operatorname{Re}(z)| \leq 1/2\}.$$

We will prove that D is a fundamental domain for G.

Theorem 2.12. *For all $z \in H$, there exists $g \in G$ such that $gz \in D$.*

Proof. Let $z \in H$ and $g \in G$. Then we have Equation (2.2). Now, let $r \in \mathbb{R}$. Since $c, d \in \mathbb{Z}$, the number of pairs $(c, d) \in \mathbb{Z}^2$ satisfying $0 \leq |cz + d| \leq r$ is finite. Hence, there exists a pair that generates the minimum value of $|cz + d|$, and then such a pair generates the maximal value of $\operatorname{Im}(gz)$. Thus, there exists $g \in G$ such that the value $\operatorname{Im}(gz)$ is maximal. Take that $g \in G$.

We know that there exists $n \in \mathbb{Z}$ such that $\operatorname{Re}(T^n gz) \in [\frac{-1}{2}, \frac{1}{2}]$. We have $T^n gz \in D$. Indeed, suppose $|T^n gz| < 1$. Then

$$\begin{aligned}\operatorname{Im}(ST^n gz) &= \operatorname{Im}\left(\frac{-1}{T^n gz}\right) = \operatorname{Im}\left(\frac{-\overline{T^n gz}}{|T^n gz|^2}\right) = -\operatorname{Im}\left(\frac{\overline{T^n gz}}{|T^n gz|^2}\right) \\ &= \frac{-\operatorname{Im}(\overline{T^n gz})}{|T^n gz|^2} = \frac{\operatorname{Im}(T^n gz)}{|T^n gz|^2} = \frac{\operatorname{Im}(gz)}{|T^n gz|^2} \\ &> \operatorname{Im}(gz),\end{aligned}$$

a contradiction, since we took $\operatorname{Im}(gz)$ maximal. Therefore, we obtain $|T^n gz| \geq 1$, and then $T^n gz \in D$. ■

Theorem 2.13. *If two points $z, z' \in D$ are congruent modulo G, then*

$$Re(z) = \pm\tfrac{1}{2} \ \ and \ \ z = z' \pm 1 \quad or \quad |z| = 1 \ \ and \ \ z' = -1/z.$$

Proof. Let $x + iy = z \in D$ and $g \in G$ such that $z' = gz \in D$. Note that, if $\mathrm{Im}(gz) < \mathrm{Im}(z)$, then $\mathrm{Im}(g^{-1}z') > \mathrm{Im}(z')$. Hence, we can suppose without loss of generality that $\mathrm{Im}(gz) \geq \mathrm{Im}(z)$. Then $|cz + d|^2 \leq 1$, which implies that

$$|cz + d| \leq 1 \tag{2.4}$$

and

$$c^2x^2 + 2cdx + d^2 + c^2y^2 \leq 1.$$

Since $z \in D$, we have that $|z| = x^2 + y^2 \geq 1$, and then

$$1 \geq c^2x^2 + 2cdx + d^2 + c^2y^2 \geq c^2 + 2cdx + d^2. \tag{2.5}$$

If $c \geq 2$, then $1 \geq 4 + 4dx + d^2 \geq 4 - 2d + d^2$ and there is no $d \in \mathbb{R}$ that satisfies it. If $c \leq -2$, then $1 \geq 4 + 4dx + d^2 \geq 4 + 2d + d^2$ and we get the same conclusion. Hence, we obtain that $|c| \leq 1$.

- Case $c = 0$: Since $\det(g) = ad - bc = 1$, we have that $d \neq 0$. By Equation (2.5), we obtain that $d = \pm 1$.

  If $d = 1$, then $a = 1$, which implies $z' = z + b$, by the action (2.1). Since $z, z' \in D$ and $z \neq z'$ (by our hypothesis), we have that $b = \pm 1$, which leads us to $x = \pm\frac{1}{2}$. If $d = -1$, the reasoning is analogous (which leads us to the first statement of the proposition).

- Case $c = 1$: By Equation (2.5), we have that $2dx + d^2 \leq 0$, which implies that $0 \leq d \leq -2x$ or $-2x \leq d \leq 0$. Since $z \in D$, we conclude that $d \in \{-1, 0, 1\}$.

  If $d = 0$ then, by Equation (2.4), we have that $|z| \leq 1$ and, since $z \in D$, we obtain that $|z| = 1 = \det(g) = -b$, that is, $b = -1$. Thus, by the action 2.1, we get $z' = gz = a - \frac{1}{z}$. Since $z' \in D$, we obtain that $|\operatorname{Re}(z')| = |a - x| \leq \frac{1}{2}$, which implies that$-1 \leq a \leq 0$ or $0 \leq a \leq 1$. Moreover, since $z \in D$, we have that $a \in \{-1, 0, 1\}$.

  We have that $a = 0$ leads us directly to the second statement of the proposition. If $a = -1$, we obtain $z = -\frac{1}{2} + iy$ and, since $|z| = 1$, we get $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \rho$. In this case, we obtain that $z' = -1 - \frac{1}{\rho} = \rho = z$, which leads us to an absurd, by our hypothesis. If $a = 1$, by an analogous argument we can show that $z = \frac{1}{2} + i\frac{\sqrt{3}}{2} = -\overline{\rho}$. In this case, we have that $z' = 1 + \frac{1}{-\overline{\rho}} = -\overline{\rho} = z$, also an absurd.

If $d = 1$ then, by Equation (2.5), we obtain $x \leq -\frac{1}{2}$, which implies $x = -\frac{1}{2}$. Also, by Equation (2.4), we get $|z + 1| \leq 1$, which implies that $y = \frac{\sqrt{3}}{2}$, that is, $z = \rho$. Now, since $\det(g) = 1$, we have that $z' = gz = \frac{a\rho+a-1}{\rho+1} = a - \frac{1}{-\overline{\rho}} = a + \rho$, and then $a = 0$ or $a = 1$.

If $a = 1$, then $z' = gz = \frac{\rho}{\rho+1} = -\rho^2 = -\overline{\rho} = \rho + 1$, which leads us to the first statement of the proposition. If $a = 0$, then $z' = gz = -\frac{1}{\rho+1} = \rho = z$, an absurd.

If $d = -1$, then we argue analogously to obtain that $z = -\overline{\rho}$ and that $a \in \{-1, 0\}$. If $a = -1$, then $z' = gz = \rho = -\overline{\rho} - 1$, which leads us to the first statement of the proposition. If $a = 0$, then $z' = gz = -\overline{\rho} = z$, an absurd.

- Case $c = -1$: By Equation (2.3), we can consider the matrix $-g \in G$ instead of $g \in G$, and then we go back to the case $c = 1$.

■

Hence, we just proved that D is a fundamental domain for G.

Remark 2.14. Let $z \in D$ and let G_z be the stabilizer of z in G. By the above theorem, $G_z = \{1\}$, except in the following three cases:

- $z = i$, in which case $|G_z| = 2$ and $G_z = \langle S \rangle$;
- $z = \rho = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$, in which case $|G_z| = 3$ and $G_z = \langle ST \rangle$;
- $z = -\overline{\rho} = e^{\pi i/3} = 1/2 + i\sqrt{3}/2$, in which case $|G_z| = 3$ and $G_z = \langle TS \rangle$.

Theorem 2.15. *The group G is generated by the matrices S and T, that is, $G = \langle S, T \rangle$.*

Proof. Let $G' = \langle S, T \rangle$ be a subgroup of G and $g \in G$. Let z_0 lying in the interior of D, that is, $z \in \text{Int}(D)$, and $z = gz_0 \in H$. Note that there exists $g' \in G'$ such that $z' = g'z \in D$. Indeed, switching from G to G' in Theorem 2.12, we can argue exactly the same way.

Hence, we have $z' = g'z = g'gz_0$, which implies that z' and z_0 are congruent modulo G. If $z' = z_0$, then $g'g \in G_{z_0}$ and, since $z_0 \in \text{Int}(D)$, we obtain $g'g = 1$, by the above remark. Thus, we get $g = (g')^{-1}$, which implies $g \in G'$. On the other hand, if $z' \neq z_0$, then the two points z' and z_0 are in the boundary of D. This is absurd, since we took $z_0 \in \text{Int}(D)$. ■

Remark 2.16. It is clear that the group $SL_2(\mathbb{Z})$ is also generated by the matrices S and T.

3 The modular functions

The importance of modular functions lies in their role as a bridge between abstract mathematics and practical applications. In number theory, it helps us to understand modular forms, which play a key role in solving problems such as Fermat's Last Theorem.

From now on, we are interested in the functions that are invariant under the action of G on H. In other words, we are interested in the quotient H/G, meaning the quotient set of H under the action of the group G. This quotient is called the moduli space of elliptic curves.

3.1 Definition and the proof

Definition 3.1. Let $k \in \mathbb{Z}$. We say that a function $f : H \to \mathbb{C}$ is weakly modular of weight $2k$ if it is meromorphic on the half plane H and verifies the relation

$$f(z) = (cz+d)^{-2k} f(gz) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \tag{3.1}$$

for all $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and all $z \in H$.

Remark 3.2. There is an argument for opting for the weight $2k$ instead of simply k. Equation (3.1) must be true for all $g \in SL_2(\mathbb{Z})$. In particular, it holds for $g = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, which leads us to $f(z) = (-1)^k f(z)$ for all $z \in H$. Hence, if k is odd, we obtain $f(z) = 0$. This means that the only weakly modular function of odd weight is the function given by $f(z) = 0$.

Now, in order to simplify the calculations, we have another way to check if a function is weakly modular, as we can see in the following result. It is the main theorem of this article, since the proof presented is original. The reader can check another approach in [1] or a proof without the finite induction technique in [2].

Theorem 3.3. *Let f be a meromorphic function in H. Then f is a weakly modular function of weight $2k$ if, and only if, it satisfies the relations*

$$f(Tz) = f(z+1) = f(z) \tag{3.2}$$

and

$$f(Sz) = f(-1/z) = z^{2k} f(z). \tag{3.3}$$

Proof. If f satisfies Definition 3.1 then, in particular, it satisfies the identities (3.2) and (3.3). Conversely, first note that

$$Tg = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} \tag{3.4}$$

and

$$Sg = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}, \tag{3.5}$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. By Theorem 2.15, every $g \in SL_2(\mathbb{Z})$ can be written as a word in S and T. we take $\{a_i\}_{i=1}^{m}$ of length m to be a minimal such word. We suppose that the statement is valid for every $g \in SL_2(\mathbb{Z})$ of length m and we will show that the statement is also valid for every $\tilde{g}$ of length $m+1$. The case $m = 1$ is trivial, by hypothesis. Observe that, for every $\tilde{g} \in SL_2(\mathbb{Z})$ of length $m+1$, there exists $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ of length m such that $\tilde{g} = Sg$ or $\tilde{g} = Tg$. Hence, we have

$$f(\tilde{g}z) = f(Tgz) = f(gz) = (cz+d)^{2k} f(z)$$

or

$$\begin{aligned} f(\tilde{g}z) &= f(Sgz) \\ &= (gz)^{2k} f(gz) \\ &= [(gz)(cz+d)]^{2k} f(z) \\ &= \left[\frac{(az+b)(cz+d)}{cz+d}\right]^{2k} f(z) \\ &= (az+b)^{2k} f(z). \end{aligned}$$

The last equality is valid because c and d can not be simultaneously null and $z = -d/c \notin H$. Therefore, due to equations (3.4) and (3.5), we have the required. ■

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