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CONNECTIONS AND IMPLICATIONS AMONG THE GELFAND-MAZUR, FUNDAMENTAL THEOREM OF ALGEBRA AND LIOUVILLE'S THEOREM

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Abstract

The study of fundamental theorems in functional analysis often highlights deep interconnections between results that, at first glance, may appear independent. In this paper, we investigate the relationships between three classical results: Gelfand-Mazur Theorem, Liouville's Theorem and Fundamental Theorem of Algebra. While previous works have explored equivalences between pairs of these theorems, a systematic study of their mutual implications remains less developed. This paper aims to systematically explore the equivalences between these three theorems, as the literature often compares only two of them at a time. Here, we seek to highlight their mutual connections simultaneously, providing a broader understanding of their interplay. Our approach builds upon established results and techniques, offering a unified perspective that sheds light on their deeper structure.

Keywords: Gelfand-Mazur, Liouville, Fundamental Theorem of Algebra, Complex Analysis

1 Introduction

The relationships between fundamental theorems in functional analysis have been widely studied in different contexts. Many classical results establish deep connections between algebraic and analytic properties of operators, often revealing surprising equivalences. In particular, the three theorems under consideration in this paper - Gelfand-Mazur Theorem, Liouville's Theorem and Fundamental Theorem of Algebra - are known to share strong connections. However, much of the existing literature tends to focus on comparisons between two of these results at a time, without fully exploring their combined interplay.

Although previous works have explored relationships between two of these theorems at a time, this paper aims to systematically investigate the interplay among all three. For instance, studies such as Almira (2005) [2] and Singh (2006) [11] have analyzed specific cases of equivalence between pairs of theorems, but a comprehensive examination of their mutual implications remains necessary. Our goal is to bridge this gap by providing a detailed analysis of how these three results are fundamentally connected, offering a framework that unifies their proofs and conceptual foundations.

To achieve this, we proceed as follows: in Section 2, we introduce key definitions and preliminary results necessary for our discussion. In Section 3, we explore the known equivalences between pairs of theorems and extend these arguments to a full three-way equivalence. In the Section 4 we study the relations in finite dimensions and its particularities. Finally, in Section 5, we discuss the implications of our findings, highlighting possible extensions and open questions for further research.

2 Key Results and Definitions

2.1 Preliminaries

To fully understand the statements and implications of these theorems, we first introduce some foundational definitions.

Definition 2.1. A **Banach algebra** is a complete normed algebra over the complex numbers, where the norm satisfies $\|xy\| \leq \|x\|\|y\|$ for all elements x and y. Moreover, if the algebra has a unity, we call it **unital algebra**.

Definition 2.2. A **bounded entire function** is an entire (complex differentiable over all of $\mathbb{C}$) function that is also bounded, meaning there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.

Definition 2.3. An **algebraically closed** field F is a field such that every non constant polynomial in $F[x]$ has roots in F.

Definition 2.4. Let F be a field and $L \subseteq F$. We say that F is an extension of L if L is a field for the operations of F restricted to L. We can also say that L is a subfield of F.

Definition 2.5. If F is an extension of L, we call the degree of this extension the dimension of F seen as a vectorial space over L. We write $[F : L]$ this number and it holds

$$[F : L] = dim_L F.$$

When $[F : L] > 1$, then we say that the extension is proper.

Proposition 2.6. *Let F be a field. Then F is algebraically closed if and only if F has no proper field extension.*

Proof. Suppose F is algebraically closed and assume by contradiction that there exists a proper extension E of F. Then, taking $\alpha \in E \setminus F$, we have that α is algebraic over F, meaning there exists a polynomial $p(x) \in F[x]$ such that $p(\alpha) = 0$. This implies $F(\alpha)$ is a proper extension of F, contradicting the assumption that F is algebraically closed.

Conversely, if F has no proper extensions, then every non-constant polynomial $p(x) \in F[x]$ must have a root in F; otherwise, we could construct a proper extension $F(\alpha)$ where α is a root of $p(x)$, contradicting our assumption. Therefore, F is algebraically closed if and only if it has no proper extensions.

□

3 The Theorems

We state the three main theorems that form the basis of this investigation.

Theorem 3.1 (Liouville's Theorem)**.** *Every bounded entire function on $\mathbb{C}$ is constant.*

Theorem 3.2 (Gelfand-Mazur Theorem)**.** *If A is a complex Banach algebra with an identity element, and every non-zero element in A is invertible, then A is isometrically isomorphic to $\mathbb{C}$.*

Theorem 3.3 (Fundamental Theorem of Algebra)**.** $\mathbb{C}$ *is algebraically closed.*

3.1 Liouville's Theorem implies Gelfand-Mazur Theorem

Definition 3.4. Let A be a complex Banach algebra with a unit element e. Given any $a \in A$ we call the **spectrum of** a the set $\{\lambda \in \mathbb{C}; \quad a - \lambda e \text{ is not invertible}\}$ and we denote it by $\sigma(a)$.

Lemma 3.5. *Let A be a complex Banach algebra with a unit element e such that every non zero element is invertible. For any $a \in A$, the spectrum $\sigma(a)$ is a singleton subset of $\mathbb{C}$.*

Proof. In a commutative complex Banach algebra A with unity e (identity element) and the property that every non-zero element has a multiplicative inverse, we want to

show that for every element $a \in A$, the spectrum of a is nonempty. Recall that the spectrum $\sigma(a)$ of an element $a \in A$ is defined as:

$$\sigma(a) = \{\lambda \in \mathbb{C}; \quad a - \lambda e \text{ is not invertible in } A\}.$$

Our goal is to show that $\sigma(a) \neq \emptyset$ for any $a \in A$. Suppose for Contradiction that the Spectrum is Empty: Assume, for the sake of contradiction, that $\sigma(a) = \emptyset$. This assumption would mean that for every $\lambda \in \mathbb{C}$, the element $a - \lambda e$ is invertible in A. The case $a = 0$ is trivial, since $\lambda = 0 \in \sigma(0)$. By a similar argument we see that $\sigma(e) = \{1\}$. Therefore, in what follows we may consider $a \neq 0$ and $a \neq e$. Since we assumed that $a - \lambda e$ is invertible for all $\lambda \in \mathbb{C}$, we can define a function $f : \mathbb{C} \to A$ by

$$f(\lambda) = (a - \lambda e)^{-1}.$$

This function f is analytic on $\mathbb{C}$, as it represents the inverse of a continuously varying element in a Banach algebra.

Observe that as $|\lambda| \to \infty$, the norm $\|a - \lambda e\|$ behaves like $|\lambda|$, which grows unboundedly. Consequently, $\|(a - \lambda e)^{-1}\| = \|f(\lambda)\| \to 0$ as $|\lambda| \to \infty$. Thus, we can conclude that f is bounded, since f is continuous in any compact disk and decaying to zero outside the disk. Since f is analytic on all of $\mathbb{C}$ and f is bounded, by Liouville's Theorem, $f(\lambda)$ must be constant. Therefore, $f(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. If $f(\lambda) = 0$ for all λ, then $(a - \lambda e)^{-1} = 0$ for all $\lambda \in \mathbb{C}$, which is impossible since $(a - \lambda e)^{-1}$ should be an invertible element of A. Thus, our assumption that $\sigma(a) = \emptyset$ is false.

Now suppose $r, s \in \sigma(a)$. Then, $a - re = a - se = 0$ and then, $r = s$. Thus, this completes the proof.

□

Theorem 3.6 (Gelfand-Mazur Theorem)**.** *Let A be a complex Banach unital algebra with identity e and in which every non-zero element is invertible. If we assume the Liouville's Theorem in A, then A is isometrically isomorphic to the complex field $\mathbb{C}$, that is, we have that the Gelfand-Mazur Theorem holds.*

Proof. Let A be a complex commutative Banach algebra with indentity, and assume that every non-zero element in A is invertible. We want to show that A is isometrically isomorphic to $\mathbb{C}$.

For an arbitrary $a \in A$, consider its spectrum, $\sigma(a)$, which is defined by

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not invertible in } A\},$$

where e is the multiplicative identity in A. Due Lemma 3.5, every element in A has a singleton spectrum, we conclude that for every $a \in A$ there exists $\lambda \in \mathbb{C}$ such that

$a = \lambda e$ and it means that $A = \mathbb{C}e$. Now it is clear that $T : \mathbb{C} \to A$ given by $T(\lambda) = \lambda e$ is an isomorphism isometric of complex algebras and it concludes the proof.

□

3.2 Conclusion

Liouville's Theorem ensures the non-emptiness of the spectrum of any element in a complex Banach algebra, as stated in Lemma 3.5. This result is essential for constructing the isomorphism between the algebra A and the field $\mathbb{C}$. Then, any result that can assure the nonemptness of the spectrum of every element in A implies directly in the Gelafand-Mazur Theorem.

3.3 Gelfand-Mazur Theorem implies Fundamental Theorem of Algebra

Theorem 3.7 (Fundamental Theorem of Algebra)**.** *If we assume the Gelfand-Mazur Theorem, then every non-constant polynomial $p(z)$ of degree $n \geq 1$ with complex coefficients has at least one root in $\mathbb{C}$, that is, we have that the Fundamental Theorem of Algebra holds.*

Proof. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, with $a_n \neq 0$ for some $n > 1$. Suppose, by contradiction, that $p(z)$ has no roots in $\mathbb{C}$. This implies that $p(z) \neq 0$ for all $z \in \mathbb{C}$.

We will now define a commutative unital Banach algebra A associated with the polynomial $p(z)$. Consider

$$A = \mathbb{C}[z]/(p(z)),$$

where the elements of A are equivalence classes $[q(z)]$ of polynomials modulo $p(z)$. Each element of A can be represented by a polynomial of degree less than n.

Define the norm on A as:

$$\|[q(z)]\|_A = \inf_{h(z)\in(p(z))} \|q - h\|_\infty,$$

where $\|\cdot\|_\infty$ is the norm of the uniform convergence defined on $\mathbb{C}[z]$. To prove the positivity of the norm $\|\cdot\|_A$, we need to show that $(p(z))$ is a closed subspace of $\mathbb{C}[z]$ in the norm $\|\cdot\|_\infty$. Let be $(f_k(z))$ a sequence in $(p(z))$ such that $\|f_k - f\|_\infty \to 0$ for some $f(z) \in \mathbb{C}[z]$. All we have to do is to prove that $f(z) \in (p(z))$. Indeed, since $f_k(z) \in (p(z))$ for every $k \in \mathbb{N}$, then for every k there exists $g_k(z) \in \mathbb{C}[z]$ such that $f_k(z) = p(z)g_k(z)$. Since $(f_k(z))$ is convergent, it follows that $(f_k(z))$ is a Cauchy sequence. Given any pair $r, s \in \mathbb{N}$ we have

$$\|f_r(z) - f_s(z)\|_\infty = \|p(z)(g_r(z) - g_s(z))\|_\infty.$$

Since $p(z) \neq 0$ for every $z \in \mathbb{C}$ we have that

$$\lim_{r,s\to\infty} \|f_r - f_s\|_\infty = 0 \quad \Longleftrightarrow \quad \lim_{r,s\to\infty} \|g_r - g_s\|_\infty = 0.$$

The result above is clear when we put $0 < c = \inf_{z\in\mathbb{C}} |p(z)|$. Since $\mathbb{C}[z]$ is a Banach Space and (g_k) is a Cauchy sequence, it follows that there exists $g \in \mathbb{C}[z]$ such that $\|g_k - g\|_\infty \to 0$. Now,

$$f = \lim_{k\to\infty} f_k = \lim_{k\to\infty} pg_k = pg \in (p(z)).$$

It is a well known fact that $(A, \|\cdot\|_A)$ is a Banach space under these assumptions, since this norm is exactly the usual norm definided in quotient closed subspaces. We will now show that this norm is submultiplicative. Indeed, for $[f(z)], [g(z)] \in A$, we have due to the definition of the norm $\|\cdot\|_A$ that given $\varepsilon > 0$ there exist polynomials $r_1(z) = p(z)q_1(z), r_2(z) = p(z)q_2(z) \in (p(z))$ such that

$$\|f(z) - r_1(z)\|_\infty < \|[f(z)]\|_A + \varepsilon$$

and

$$\|g(z) - r_2(z)\|_\infty < \|[g(z)]\|_A + \varepsilon.$$

Thus

$$\begin{aligned}
\|[f(z)] \cdot [g(z)]\|_A &= \inf_{Q(z)\in\mathbb{C}[z]} \|f(z)g(z) - p(z)Q(z)\|_\infty \\
&\leq \|f(z)g(z) - f(z)r_1(z) - g(z)r_2(z) + r_1(z)r_2(z)\|_\infty \\
&= \|(f(z) - r_1(z))(g(z) - r_2(z))\|_\infty \\
&\leq \|f(z) - r_1(z)\|_\infty \cdot \|g(z) - r_2(z)\|_\infty \\
&< (\|[f(z)]\|_A + \varepsilon) \cdot (\|[g(z)]\|_A + \varepsilon).
\end{aligned}$$

Due to the arbitrariness of $\varepsilon > 0$, we have

$$\|[f(z)] \cdot [g(z)]\|_A \leq \|[f(z)]\|_A \cdot \|[g(z)]\|_A.$$

Thus, $(A, \|\cdot\|_A)$ is a commutative Banach Algebra with identity (the constant polynomial 1). The last thing to do before applying the Gelfand-Mazur Theorem is to prove that every nonzero element in A has a multiplicative inverse. Without loss of generality we can consider that p is irreducible, otherwise we can we can apply this result to every irreducible factor of p and we derive the same results. Since p is irreducible over $\mathbb{C}[z]$, then $(p(z))$ is maximal and thus, A is a field and therefore, every nonzero element has

a mutiplicative inverse. Now, the algebra A satisfies all the conditions of the Theorem of Gelfand Mazur wich guarantees that

$$A = \mathbb{C}[z]/(p(z)) \cong \mathbb{C}.$$

Since

$$\dim_{\mathbb{C}} A = deg(p) = 1 \quad \text{(because } p \text{ is irreducible)}$$

we have that $p(z) = az + b$, where $a, b \in \mathbb{C}$. This shows that the polynomial $p(z)$ either has a root in $\mathbb{C}$ or $p(z)$ is constant. In both cases, we reach a contradiction with the hypothesis that the degree of $p(z)$ is initially greater than 1. Then, p must have at least one root in $\mathbb{C}$.

□

3.4 Conclusion

We prove the Fundamental Theorem of Algebra by showing that $\mathbb{C}$ has no proper field extension. We also see that Liouville's Theorem can assure this result since the Lemma 3.5 is enough to prove it. There is another proof of the FTA using Liouville's Theorem, but it approaches directly by complex analysis arguments about a given polynomial.

3.5 FTA implies Liouville's Theorem

Lemma 3.8. *Let be f an entire function with representation*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

for all $z \in \mathbb{C}$. Given N natural, let be P_N the polynomial

$$P_N(z) = f(z) - \sum_{n>N} a_n z^n,$$

for every $z \in \mathbb{C}$. If P_N is constant for every N greater than some $n_0 \in \mathbb{N}$, then f is constant.

Proof. We have

$$f(z) = P_N(z) + \sum_{n>N} a_n z^n$$

for every N and all $z \in \mathbb{C}$. By hypothesis

$$f'(z) = \left(P_N(z) + \sum_{n=N+1}^{\infty} a_n z^n \right)' = \sum_{n=N+1}^{\infty} n a_n z^{n-1},$$

for every $z \in \mathbb{C}$ and $N > n_0$. Thus, if we take the limit as $N \to \infty$ we have $f'(z) = 0$ for all $z \in \mathbb{C}$, because f' is entire and thus it series converges uniformly. Since $f' \equiv 0$, we have f constant as we desired.

□

Theorem 3.9 (Liouville's Theorem)**.** *If we assume the Fundamental Theorem of Algebra, then every bounded entire function on $\mathbb{C}$ is constant, that is, we have that the Liouville's Theorem holds.*

Proof. Let be f a bounded and entire function. So, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

for all $z \in \mathbb{C}$. Fixed N, let be P_N the polynomial defined by

$$P_N(z) = f(z) - \sum_{n>N} a_n z^n, \quad z \in \mathbb{C}.$$

Then,

$$f(z) - P_N(z) = \sum_{n>N} a_n z^n,$$

for all $z \in \mathbb{C}$. Suppose f non constant. Then, due Lemma 3.8 (contrapositive) there are $N_1 < N_2 < N_3 < \ldots \in \mathbb{N}$ such that P_{N_j} is non constant for every $j \in \mathbb{N}$. Due the Fundamental Theorem of Algebra, for every j there is $z_j \in \mathbb{C}$ such that $P_{N_j}(z_j) = 0$. Thus,

$$f(z_j) = \sum_{n>N_j} a_n z_j^n,$$

for every $j \in \mathbb{N}$. It assures that every non constant and entire function must admit zero as its limit point, because

$$\lim_{j\to\infty} f(z_j) = \lim_{j\to\infty} \sum_{n>N_j} a_n z_j^n = 0.$$

Since f is bounded there exists $M > 0$ such that $|f(z)| \leq M$, for all $z \in \mathbb{C}$. Let be $w \in \mathbb{C}$ such that $|w| > 2M$. If we put $g(z) = f(z) + w$ for every $z \in \mathbb{C}$, we have

$$|g(z)| \geq |w| - |f(z)| \geq 2M - M = M > 0.$$

Since f is non constant and w is fixed, then g is entire and non constant. Then, by the reasoning above we should have that 0 must be a limit point of $g(\mathbb{C}) = \{g(z); \quad z \in \mathbb{C}\}$. But since $|g(z)| \geq M$ for all $z \in \mathbb{C}$ we see that 0 can not be a limit point of $g(\mathbb{C})$. This contradiction shows that f must be constant to satisfy all the conditions in the theorem.

□

4 In finite dimension The Fundamental Theorem of Algebra implies Gelfand-Mazur Theorem

We will prove now that if the Algebra A is finite dimensional over $\mathbb{C}$, then we can derive directly the Gelfand-Mazur Theorem by using the Fundamental Theorem of Algebra.

Theorem 4.1 (Gelfand-Mazur Theorem finite dimensional case)**.** *If A is a finite dimensional complex Banach algebra with an identity element, and every non-zero element in A is invertible, then A is isometrically isomorphic to $\mathbb{C}$.*

Proof. The only change to be done in the proof of the Gelfand-Mazur Theorem (classic statement) is about the nonemptness of the spectrum of the elements of A. So we will only prove that given $a \in A$ we have $\sigma(a) \neq \emptyset$.

For every $a \in A$ consider $T_a : A \to A$ given by

$$T_a(x) = ax, \quad \forall\, x \in A.$$

It is clear that T_a is an isomorphism of vector spaces for all $a \neq 0$ and when $a = 0$ we have the null operator. Furthermore, let $\|\cdot\|_A$ stands for the norm on A and $\|\cdot\|_{A,A}$ for the usual norm on $\mathcal{L}(A, A)$. Given $a, b \in A$ and $\lambda \in \mathbb{C}$, we have

$$T_{a+\lambda b}(x) = (a + \lambda b)x = ax + \lambda bx = T_a(x) + \lambda T_b(x), \quad \forall x \in A,$$

and

$$\|T_a\|_{A,A} = \sup_{\|x\|_A=1} \frac{\|T_a(x)\|_A}{\|x\|_A} = \sup_{\|x\|_A=1} \|ax\|_A \leq \|a\|_A.$$

Since for $x = 1$ we attain the supremum, we have $\|T_a\|_{A,A} = \|a\|_A$ for every $a \in A$. So, we can write $A \hookrightarrow \mathcal{L}(A, A)$ and this embeding is linear and isometric.

Due to the Fundamental Theorem of Algebra every $T_a \in \mathcal{L}(A, A)$ has an eigenvalue since its characteristic polynomial has roots. Let us show that $\sigma(T_a) \neq \emptyset \implies \sigma(a) \neq \emptyset$. Let be $a \in A$ and $\lambda \in \sigma(T_a)$. Thus, there exists $x \neq 0$ such that $ax = \lambda x$. That implies $a = \lambda \in \mathbb{C}$. In this case we have $a \in \sigma(a)$ since $a - ae = 0$. Finally, for every $a \in A$ we have $\sigma(a) = \{a\}$ because every nonzero element in A is invertible and the singletonesss follows in the same way as in the proof of the classic statement of Gelfand- Mazur Theorem.

□

4.1 Conclusion

We used the FTA to prove that in the case when A is a finite dimensional complex algebra fullfiling the conditions of the Gelfand-Mazur Therem this theorem still holds and we do not need to invoke Liouville's Theorem. The converse is always true as we have shown above. Then, for finite dimensions, they are equivalents.

The main idea in this approach is that the FTA guarantees that the spectrum of any element of this algebra is nonempty. All the rest of the ideas are equal to the classic proof. If we could gauarantee that even in infinite dimension we have $\sigma(T_a) \neq \emptyset$ we are proving that both theorems are equivalents in any case.

5 Conclusion of this Work

In conclusion, this work has explored the intricate connections among the Fundamental Theorem of Algebra (FTA), Liouville's Theorem, and the Gelfand-Mazur Theorem, demonstrating how these fundamental results intersect. We showed that the Gelfand-Mazur Theorem implies the FTA and, within finite-dimensional algebras, the two are equivalent. Additionally, we established that the Gelfand-Mazur Theorem implies Liouville's Theorem, linking boundedness in complex analysis to algebraic principles, and that Liouville's Theorem can be derived independently of Cauchy's Integral Formula.

The FTA alone is sufficient to prove Liouville's Theorem, completing the cycle of implications, though the result does not hold over $\mathbb{R}$ due to polynomials P_N potentially lacking roots, as we can see in the following that confirm the equivalence of the three theorems in the context of real numbers. We will show through natural examples that the algebraic completeness of $\mathbb{C}$ is indispensable for the statements of the Gelfand-Mazur and Liouville's theorems to hold.

Example 5.1 (Liouville's Theorem in $\mathbb{R}$)**.** Let $A_{\mathbb{R}}$ be the set of all bounded functions $f : \mathbb{R} \to \mathbb{R}$ that admit a Taylor series representation at every point of the real line with an infinite radius of convergence.

If Liouville's Theorem were valid for the set of real numbers, then the function $g(x) = \sin x$ would have to be constant since it is bounded and analytic in the real line by Taylor's Theorem. But if it was constant, then the same should happen to its derivative, given by $g'(x) = \cos x$. But, as we have learned $\cos 0 = 1 \neq 0 = \cos \frac{\pi}{2}$. Thus, Liouville's Theorem does not hold in the real line.

Example 5.2 (Gelfand-Mazur Theorem in $\mathbb{R}$)**.** Consider $\mathbb{C}$ with its usual operations and properties.

The set $\mathbb{C}$ is a real vector space with dimension 2. Note that with the usual norm of $\mathbb{C}$ we find that $\mathbb{C}$ is a real Banach Algebra, with unity and it is commutative. Moreover, every nonzero element has a multiplicative inverse. Thus, if Gelfand-Mazur Theorem were valid in the real line we should have that $\mathbb{C}$ and $\mathbb{R}$ were isomorph. But it is impossible, since they have different dimensions over $\mathbb{R}$.

Remark 5.3. It is worth highlighting that the algebraic completeness of $\mathbb{C}$ is equivalent to ensuring that no other distinct and non-isomorphic field can extend it finitely. This statement reduces to showing that the algebra of operators of the supposed field satisfies the condition that every element has a nonempty spectrum since the algebraic completeness of $\mathbb{C}$ implies that every operator in this field has eigenvalues. This is the essence of the connection between the Fundamental Theorem of Algebra and the Gelfand-Mazur Theorem.

These findings highlight the foundational role of the Gelfand-Mazur Theorem in complex analysis and algebra and its broader implications. By illustrating the power of abstract algebraic methods in understanding analytic properties, this work invites further exploration of these classical results and their applications.

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