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ON TUBES IN $\mathbb{R}^{N+1}$

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Abstract

In this paper, we introduce the notion of tube with arbitrary cross sections around a curve for which we calculate the volume and give a generalization for the second theorem of Pappus. The first Theorem of Pappus is obtained for sphere tubes in arbitrary dimension.
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1 Introduction

The calculation of the volume of disk tubes in $\mathbb{R}^{n+1}$ and in the euclidean sphere $\mathbb{S}^n$ appears in [8]. Actually, the main goal of this paper is to work within the sphere. In [5], the author generalizes the Pappus theorems for genuine tubes around an arbitrary curve in $\mathbb{R}^3$. In the case of the second Pappus theorem, his proof makes use of the divergence theorem. Here, we will establish such a theorem for genuine tubes in $\mathbb{R}^{n+1}$ by using a different strategy, namely, the formula for change variable in multiple integrals.

The first theorem of Pappus concerns the calculation of surface area, while the second concerns the calculation of volume of solids of revolution. They are as follows.

Theorem A (The First Theorem of Pappus). *If a plane curve $\mathcal{C}_P$ of length M is rotated about an axis that does not meet $\mathcal{C}_P$, then S, the area of the surface generated, is given by*

$$S = ML$$

where L is the length of the curve described by the centroid of $\mathcal{C}_P$ during the rotation.

Theorem B (The Second Theorem of Pappus). *Let $\mathcal{D}$ be a region in the plane and let $\mathcal{L}$ be a line in the plane of $\mathcal{D}$. If $\mathcal{L}$ does not meet $\mathcal{D}$, then the volume of the solid generated when $\mathcal{D}$ is rotated around $\mathcal{L}$ is given by*

$$V = A\mathcal{L},$$

where A is the area of $\mathcal{D}$ and $\mathcal{L}$ is the perimeter of the circle described by the centroid of $\mathcal{D}$.

Probably, the torus of revolution $T(a, b) \subset \mathbb{R}^3$, generate by revolving the circle

$$(x-b)^2 + z^2 = a^2, \quad y = 0,$$

around the z-axis, is the best known example of tube around a curve f in $\mathbb{R}^3$. This curve, which is called the axis of $T(a, b)$, is given by $f(t) = (b\cos t, b\sin t, 0)$, $0 \leq t \leq 2\pi$, whose trace is the circle $x^2 + y^2 = b^2$ in the xy-plane. $T(a, b)$ is a regular surface, whenever $0 < a < b$ and

$$X(v, t) = ((b + a\cos v)\cos t, (b + a\cos v)\sin t, a\sin v),$$

for (v, t) ranging over $[0, 2\pi] \times [0, 2\pi]$, is onto $T(a, b)$. Of course the restriction of X to the open rectangle $(0, 2\pi) \times (0, 2\pi)$ is a parametrization of $T(a, b)$. A good way of rewriting this map consists in replacing $(a\cos v, a\sin v)$ by (x_1, x_2) to obtain a new map

$$G(x_1, x_2, t) = ((b + x_1)\cos t, (x_1 + b)\sin t, x_2), \subset \mathbb{R}^3,$$

$(x_1, x_2, t) \in S^1(a) \times [0, 2\pi]$, where $S^1(a)$ is the circle $x_1^2 + x_2^2 = a^2$ of the x_1x_2-plane of the tridimensional space $\mathbb{R}^3$ with euclidean coordinates (x_1, x_2, t). This map sends the right circular cylinder over S^1 having height 2π onto $T(a, b)$, transforming each copy of S^1 in the cylinder onto a copy of the generator circle of $T(a, b)$. By composing G with $h(v, t) = (a\ \cos v, a\ \sin v, t)$ we can recover X. At this moment, it is convenient to observe that $\widetilde{G}$, the extension of G to the solid cylinder $B^{[2]}[a] \times [0, 2\pi]$, fills the solid enclosed by $T(a, b)$, where $B^{[2]}[a]$ denotes the compact disk of radius a of the x_1x_2-plane. $\widetilde{G}$ becomes, for $(x_1, x_2, t) \in B^{[2]}[a] \times [0, 2\pi] \subset \mathbb{R}^3$,

$$\widetilde{G}(x_1, x_2, t) = ((b + x_1)\cos t, (x_1 + b)\sin t, x_2), \tag{1.1}$$

which is onto the solid torus $\overline{T(a, b)}$. The next picture shows X, G and the torus $T(a, b)$ together with the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of its axis $f(t) = (b\cos t, b\sin t, 0)$. It is not hard to see that

$$\begin{aligned}
\mathbf{T}(t) &= (-\sin t, \cos t, 0),\\
\mathbf{N}(t) &= (-\cos t, -\sin t, 0),\\
\mathbf{B}(t) &= (0, 0, 1).
\end{aligned}$$

Furthermore, $\kappa = 1/b$ and $\tau = 0$ are the curvature and torsion of f. In order to generalize the ideas in the exposition above, we verify easily that G ($\widetilde{G}$) coincides with

$$f(t) - x_1\,\mathbf{N}(t) + x_2\,\mathbf{B}(t).$$

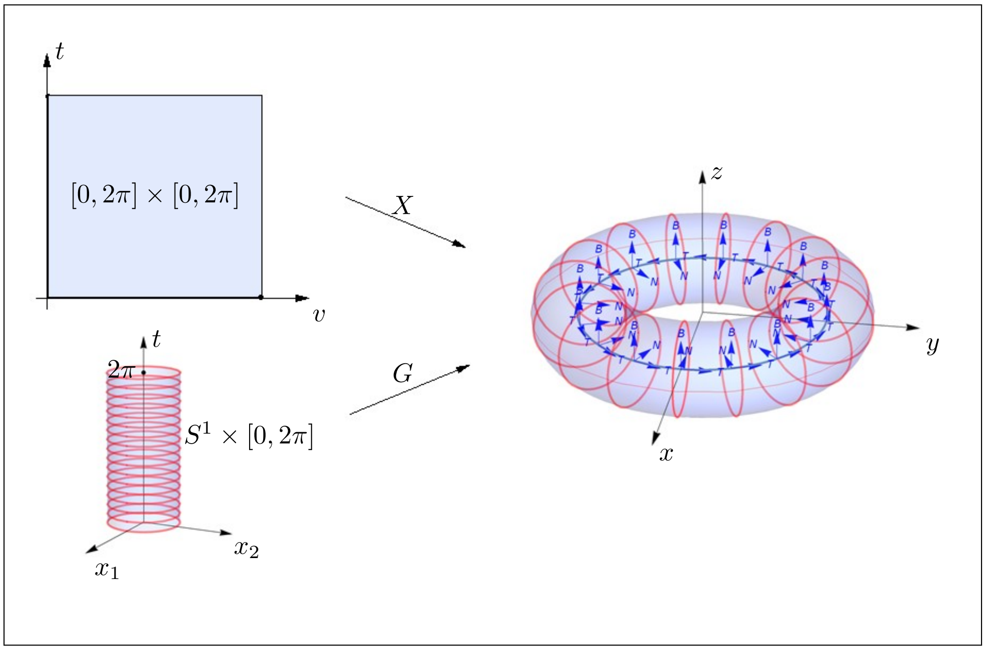

Figure 1. **Torus of revolution**

The minus signal was included because $\{-\mathbf{N}, \mathbf{B}\}$ has the same orientation as $\{e_2, e_3\}$ in the revolving x_1x_2-plane. However, we can choose with no additional difficulties

$$f(t) + x_1\,\mathbf{N}(t) + x_2\,\mathbf{B}(t) = ((b - x_1)\cos t, (b - x_1)\sin t, x_2)$$

to generate the same torus. From this model, we will construct tubes of either spheres or disks or any other nice region around a n-regular curve f in $\mathbb{R}^{n+1}$. Going back to (1.1) and taking the rule of $\widetilde{G}$, we define the following map, defined in Ω, by

$$H(x_1, x_2, t) = ((b + x_1)\cos t, (x_1 + b)\sin t, x_2),$$

where Ω is the open set $B^{[2]}(\widetilde{a}) \times (0, 2\pi)$ and $B^{[2]}(\widetilde{a})$ is the open disk of radius $\widetilde{a}$, for some $\widetilde{a}$ such that $b > \widetilde{a} > a$. It is not hard to see that H is injective and the absolute values of the its jacobian determinant is

$$\begin{aligned} |\det JH| &= \left|\det\left(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \frac{\partial H}{\partial t}\right)\right| \\ &= \left\|\frac{\partial H}{\partial x_1} \wedge \frac{\partial H}{\partial x_2} \wedge \frac{\partial H}{\partial t}\right\| = |b + x_1| = b + x_1 > 0, \end{aligned}$$

since $-b < -\widetilde{a} < x_1 < \widetilde{a} < b$. Thus, for a sufficiently small $\epsilon > 0$ we can calculate the volume of $\overline{T}_\epsilon \subset \overline{T(a,b)}$, the image under H of the compact set $B^{[2]}[a] \times [\epsilon, 2\pi - \epsilon]$, which, by using the change of variable in multiple integrals, is

$$\begin{aligned}
\operatorname{vol} \overline{T}_\epsilon &= \iiint_{\overline{T}_\epsilon} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int_\epsilon^{2\pi-\epsilon} \left(\iint_{B^{[2]}[a]} (b + x_1) \mathrm{d}x_1 \mathrm{d}x_2 \right) \mathrm{d}t \\
&= b \int_\epsilon^{2\pi-\epsilon} \left(\iint_{B^{[2]}[a]} \mathrm{d}x_1 \mathrm{d}x_2 \right) \mathrm{d}t \\
&= 2\pi a^2 b(\pi - \epsilon),
\end{aligned}$$

where we have used $\iint_{B^{[2]}[a]} x_1 \mathrm{d}x_1 \mathrm{d}x_2 = 0$, fact that is intimately connected to the barycenter[1] of the disk $D[a]$, namely its center $(0, 0)$. Hence

$$\operatorname{vol} \overline{T(a,b)} = \lim_{\epsilon \to 0} \operatorname{vol} \overline{T}_\epsilon = 2\pi^2 a^2 b,$$

which is a well known result and that can be obtained from the second theorem of Pappus applied to the torus: the volume of $\overline{T(a,b)}$ equals the area of $B^{[2]}[a]$ times the length of the curve described by the center of the revolving disk.

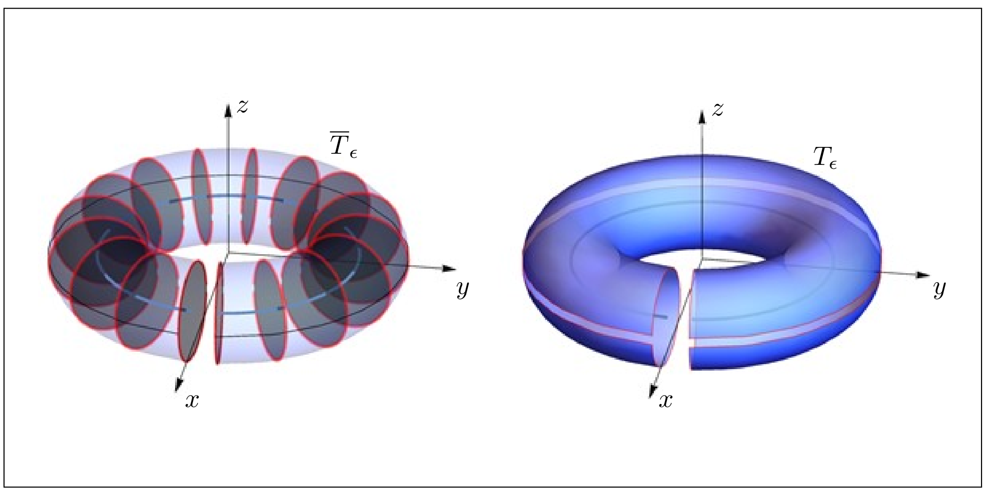

Figure 2. Calculating the volume and the area of Torus

By the same reasoning, we can find the area of the $T(a, b)$, by working with the restriction of X to the rectangle $R_\epsilon = [\epsilon, 2\pi - \epsilon] \times [\epsilon, 2\pi - \epsilon]$ whose image is $T\epsilon$, as in the picture above (see [3]-**2.5-Example 5**). The calculations are as follows.

[1] The barycenter (see also (3.4), page 215) of a compact region $K \subset \mathbb{R}^2$ with area > 0 is the point $B_K = \frac{1}{\operatorname{area} K}(\iint_K x \mathrm{d}x\, \mathrm{d}y, \iint_K y \mathrm{d}x\, \mathrm{d}y)$.

$$
\begin{aligned}
\operatorname{area} T_\epsilon &= \iint_{R_\epsilon} \left\| \frac{\partial X}{\partial t} \wedge \frac{\partial X}{\partial v} \right\| \mathrm{d}t \, \mathrm{d}v = \iint_{R_\epsilon} \sqrt{\left\| \frac{\partial X}{\partial t} \right\|^2 \left\| \frac{\partial X}{\partial v} \right\|^2 - \left(\frac{\partial X}{\partial v} \cdot \frac{\partial X}{\partial v} \right)^2} \, \mathrm{d}t \, \mathrm{d}v \\
&= \int_\epsilon^{2\pi - \epsilon} \left(\int_\epsilon^{2\pi-\epsilon} (b + a \cos v) \, \mathrm{d}v \right) \mathrm{d}t \\
&= 4a^2 \epsilon \sin \epsilon - 4\pi a^2 \sin \epsilon + 4ab\epsilon^2 - 8\pi ab\epsilon + 4\pi^2 ab,
\end{aligned}
$$

whose limit, as ϵ tends to zero, is $4\pi^2 ab$, which is the area of $T(a,b)$. Now, we have the length of $S^1(a)$ times the length of curve described by the center of the revolving circle, which comes from the first theorem of Pappus. The cited theorems of Pappus concern either surfaces or solids of revolution generated by either a simple plane curve or the region enclosed by it. Next, we will deal with these theorems in a more general sense in any dimension. A generalization of them, in the tridimensional case, appears in [5].

2 Constructing tubes

Given an interval J, let Ω_J be the set

$$\Omega_J(S_t) = \cup_{t \in J} S_t \subset \mathbb{R}^{n+1},$$

where

$$S_t = \{(x_1, x_2, \ldots, x_n, t); \quad (x_1, x_2, \ldots, x_n) \in \widetilde{S}_t \subset \mathbb{R}^n\}$$

and $\widetilde{S}_t$, the projection of S_t in $\mathbb{R}^n$, has positive volume. So, Ω_J is the solid whose intersection with the hyperplane $x_{n+1} = t$ is S_t, for each $t \in J$. When all projections coincide with a certain region S, $\Omega_J(S)$ is a *solid cylinder* with cross section S, case in which we write $S_t = (S,t)$ and $\operatorname{vol} \widetilde{S}_t = \operatorname{vol} S > 0$, of course measured in $\mathbb{R}^n$. Now, given $I = [a,b] \subset J$ and $F : \Omega_J(S_t) \longrightarrow \mathbb{R}$ continuous, we suppose that the integral

$$\int_{\Omega_I(S_t)} F(X,t) \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \, \mathrm{d}t = \int_{\Omega_I(S_t)} F(x_1, x_2, \ldots, x_n, t) \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \, \mathrm{d}t$$

always exists. Thus, by using the Fubini theorem, we get

$$\int_{\Omega_I(S_t)} F(X,t) \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \, \mathrm{d}t = \int_a^b \left(\int_{\widetilde{S}_t} F(X,t) \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \right) \mathrm{d}t.$$

In particular, $\operatorname{vol} \Omega_J(S_t) = \int_a^b \operatorname{vol} \widetilde{S}_t \, \mathrm{d}t$.

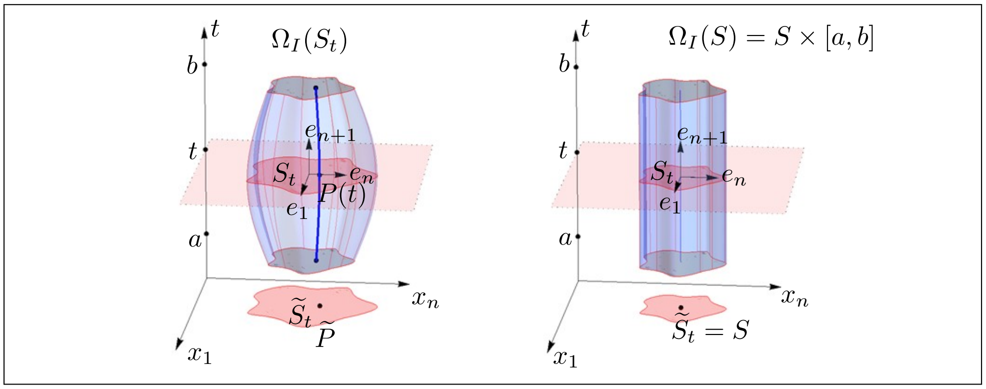

Figure 3. Solids with cross section S_t

Let $f : J \longrightarrow \mathbb{R}^{n+1}$ be a regular curve with Frenet referential

$$\{ V_1(t),\, V_2(t), \ldots,\, V_{n+1}(t) \}$$

and $\Omega_J(S_t)$ as above. We construct a tube around f, which we call the *tube generated by* $\Omega_J(S_t)$ as follows. Choose in each slice $\mathbb{R}^n \times \{t\}$ one point, say

$$P = P(t) = (p_1(t), p_2(t), \ldots, p_n(t), t)$$

(in general, we choose $P(t)$ in S_t) and put S_t in the hyperplane orthogonal to the trace of f at $f(t)$, making the point $P(t)$ stay on $f(t)$ and each e_j of the canonical basis of $\mathbb{R}^n$ on V_{j+1}. This process ends with what we call *the tube around* f *generated by* $\Omega_J(S_t)$ *and centered at* $P(t)$, which will be indicated by $\Gamma_P(f, \Omega_J(S_t))$ or simply Γ with little danger of confusion. Intuitively, everything happens as if a region S is moving along a curve by expanding or shrinking and always stuck to the curve by a point P. We also say that the curve f is the *axis of the tube*.

$\Gamma_P(f, \Omega_J(S_t))$ can be obtained as image of $\Omega_J(S_t)$ under the map $G : \mathbb{R}^n \times J \longrightarrow \mathbb{R}^{n+1}$ given by

$$G(X, t) = f(t) + \sum_{j=1}^{n} (x_j - p_j(t))\, V_{j+1}(t), \tag{2.1}$$

where $(X, t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^{n+1}$. Of course that this map changes with a change in the choice $\widetilde{P}(t) = (p_1(t)\ p_2(t)\ \ldots\ p_n(t))$. Geometrically, G sends each cross-sections S_t onto a copy of it, also called *cross section of the tube*, into the hyperplane passing through $f(t)$ and parallel to the subspace generated by $V_2,\ V_3,\ \ldots,\ V_{n+1}$, gluing $P(t)$

and $f(t)$ and laying e_j, the j-th element of the canonical basis, on V_{j+1}, for j running from 1 to n, as we see in the Figure 4 below.

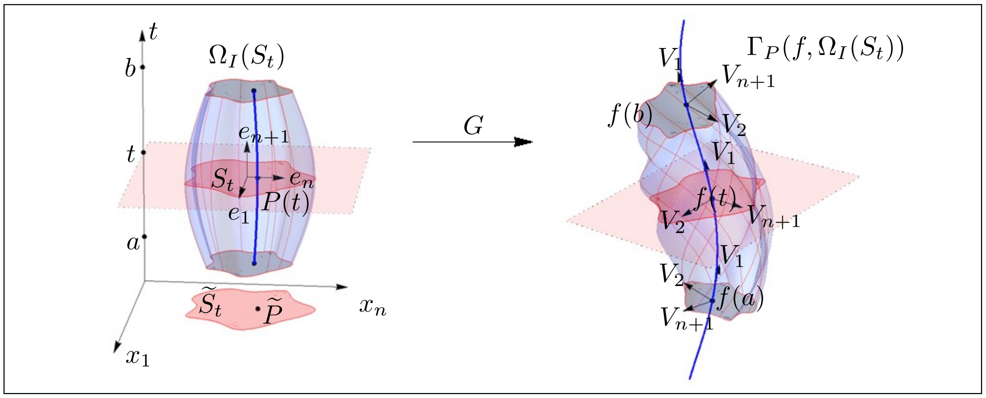

Figure 4. Tube around the curve f

Remark 2.1. Actually, in (2.1), we could use $\widetilde{F}_t(V_{j+1}(t))$ instead $V_{j+1}(t)$, where $\widetilde{F}_t$ is a differentiable family of linear isometries of $W_t = \text{span}\{V_j(t),\ 2 \le j \le n+1\}$ induced by a differentiable family of linear isometry of $\mathbb{R}^n$, F_t, producing so a *twisted tube*. If $L_t : \mathbb{R}^n \longrightarrow W_t$ is the linear isometry such that $L_t(e_j) = V_{j+1}(t)$, then $\widetilde{F}_t$ is constructed as in the commutative diagram below. Note that when each cross section S_t of $\Omega_J(S_t)$ is a disk centered at the $(0, 0, ..., t)$, then the disk tube constructed from a family of isometries F_t must coincide with that obtained from G in (2.1), which is constructed by using the identity map. In fact, disks centered at origin are invariant under linear isometries.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{L_t} & W_t \\ {\scriptstyle F_t}\uparrow & & \downarrow{\scriptstyle \widetilde{F}_t} \\ \mathbb{R}^n & \xleftarrow[L_t^{-1}]{} & W_t \end{array}$$

Of course a tube $\Gamma = \Gamma_P(f, \Omega(S_t))$ may have self-intersections, which depend on the curvatures of f and the size of the S_t. If Γ_P has no such a self-intersection, that is, if G is injective, Γ is called a *genuine tube*. In the picture above, we have a *genuine tube*. In the cases where the cross section of $\Omega_J(S_t)$ are disks, we say that Γ is a *disk tube* around f. Below, in the Figure 5, we see two cases of self-intersections in *disk tubes* around the same curve. In both cases, a reduction in the sizes of the cross sections would produce genuine disk tubes. The next theorem guarantees the existence of genuine disk tubes.

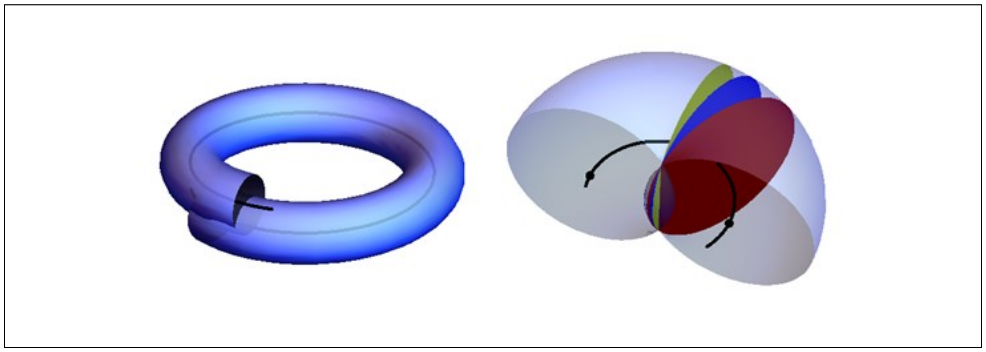

Figure 5. **Two nongenuine tubes**

Theorem 2.2. *Let* $f : J \longrightarrow \mathbb{R}^{n+1}$ *be a* n*-regular curve with speed* ν*, Frenet apparatus*

$$\mathcal{A} = \{\kappa_1, \ldots, \kappa_{n-1}, V_1, \ldots, V_n\},$$

and consider a compact interval $I = [a, b] \subset J$*. If* f *is injective, then there exists* $r > 0$ *such that the map* G *in (2.1), with* $p_j(t) = 0$*,* $1 \leq j \leq n$*, carries* $B^{[n]}[r] \times I$ *onto the compact genuine disk tube* $\Gamma = G(B^{[n]}[r] \times I)$*, where* $B^{[n]}[r]$ *is the compact disk of radius* r *in* $\mathbb{R}^n$*.*

Proof. We start by calculating the absolute value of the jacobian determinant of G, that is given by

$$|\det JG| = \left|\det\left(\frac{\partial G}{\partial x_1}\,\frac{\partial G}{\partial x_2}\cdots\frac{\partial G}{\partial x_n}\,\frac{\partial G}{\partial t}\right)\right| = \left\|\frac{\partial G}{\partial x_1} \wedge \frac{\partial G}{\partial x_2} \wedge \cdots \wedge \frac{\partial G}{\partial x_n} \wedge \frac{\partial G}{\partial t}\right\|.$$

From $G(x_1, x_2, \ldots, x_n, t) = f(t) + \sum_{j=1}^{n} x_j\, V_{j+1}$, we get that $\frac{\partial G}{\partial x_j} = V_{j+1}$, for $1 \leq j \leq n$, and that

$$\begin{aligned}\frac{\partial G}{\partial t} &= f' + \sum_{j=1}^{n} x_j\, V'_{j+1}\\ &= \nu\left(V_1 + \sum_{j=1}^{n-1} x_j(-\kappa_j\, V_j + \kappa_{j+1}\, V_{j+2}) - x_n\, \kappa_n\, V_n\right)\\ &= \nu(1 - x_1\, \kappa_1)\, V_1 + \nu \sum_{j=2}^{n+1} z_j\, V_j,\end{aligned}$$

for some coefficients z_j, which depend on x_k, $1 \leq k \leq n$, and κ_m, for $2 \leq m \leq n$. Hence,

$$\left\| \frac{\partial G}{\partial x_1} \wedge \frac{\partial G}{\partial x_2} \wedge \cdots \wedge \frac{\partial G}{\partial x_n} \wedge \frac{\partial G}{\partial t} \right\| = \nu |1 - x_1 \kappa_1| \, \| V_2 \wedge \cdots \wedge V_{n+1} \wedge V_1 \| = \nu |1 - x_1 \kappa_1|$$

and, thus, the absolute value of the jacobian determinant of G equals $\nu|1 - x_1 \kappa_1|$. In particular, fixed $t \in J$, and letting $P = (0, 0, \ldots, 0, t)$, we obtain that $|\det JG(P)| = \nu > 0$. Since, f is injective, it follows that G is injective on the compact set

$$K = \{(0, 0, \ldots, 0)\} \times [a, b].$$

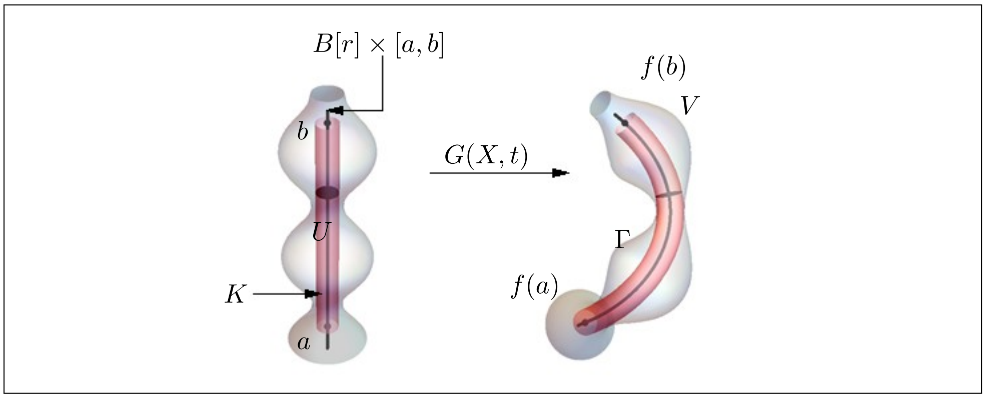

Figure 6. Illustration for the Theorem 2.2

Hence, by using the generalized version of the inverse function theorem, we get an open set $U \supset K$ and an open set $V \supset G(K) = f([a, b])$ such that the restriction $G : U \longrightarrow V$ is a diffeomorphism. Since U is an open set, there is a family of small open cylinders $B^{[n]}(\epsilon_\lambda) \times (a_\lambda, b_\lambda)$ whose union covers K. From this family, we get a finite number of such cylinders, say $B^{[n]}(\epsilon_j) \times (a_j, b_j)$, $1 \leq j \leq m$, such that $K \subset \cup_{j=1}^{m} \left(B^{[n]}(\epsilon_j) \times (a_j, b_j) \right)$. Now, $\Gamma = G(B^{[n]}[r] \times [a, b])$, $r < \min\{\epsilon_j\}$, becomes a genuine compact disk tube, as we see in the Figure 6. □

Remark 2.3. Many times, we have curves $f : J \longrightarrow \mathbb{R}^n$ such that its trace is contained in an affine subspace, say, $W = X_0 + S$, where X_0 is a point and S is a m-dimensional subspace of $\mathbb{R}^n$, $m < n$ (think of f as a plane curve in $\mathbb{R}^3$). In many of these cases, when f is $(m-1)$-regular, we may construct a partial Frenet apparatus for f, obtaining

$$\mathcal{A} = \{ \kappa_1 > 0, \ldots, \kappa_{m-1} > 0, \, \kappa_m = 0, \, V_1, \ldots, V_m, \, V_{m+1} \},$$

which satisfies the Frenet equations and, moreover, for each $t \in J$, the subspace generated by $\{ V_1, V_2, \ldots, V_m \}$ equals S and $V_{m+1} \in S^\perp$ is constant. Then, we consider an orthonormal basis for the orthogonal complement of S, say, $\{V_{m+1}, V_{m+2}, \ldots, V_n\}$ and define $\kappa_{m+1}, \ldots, \kappa_{n-1}$ to be all zero. So, we have a full Frenet apparatus:

$$\mathcal{A} = \{\kappa_1, \ldots, \kappa_{n-1}, V_1, \ldots, V_n\},$$

where $\{V_{m+1}, V_{m+2}, \ldots, V_n\}$ is constant and $\kappa_j = 0$, $m \leq j \leq n-1$. Moreover, the Frenet equations remain true for this extended apparatus.

The apparatus of the straight line $f(t) = X_0 + t\,V_1$, $\|V_1\| = 1$, is now defined and is given by

$$\mathcal{A} = \{\kappa_1, \ldots, \kappa_{n-1}, V_1, \ldots, V_n\},$$

where the curvatures are zero and $\{V_1, \ldots, V_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Also, we may consider the regular curves in $\mathbb{R}^m$, $m < n$, as curves in $\mathbb{R}^n$, by filling with zeros after the m-th coordinate: $f(t) = (f_1(t)\ f_2(t)\ \ldots\ f_m(t), 0, 0, \ldots, 0)$, case in which the Frenet apparatus becomes

$$\mathcal{A} = \{\kappa_1, \ldots, \kappa_{n-1}, V_1, \ldots, V_n\},$$

where $\{V_{m+1}, V_{m+2}, \ldots, V_n\}$ is part of the canonical basis of $\mathbb{R}^n$ and $\kappa_j = 0$, $m \leq j \leq n-1$. Of course V_j, $1 \leq j \leq m$, are those vector fields of the original curve with zeros on the last $n-m$ coordinates. For example, the curve in $\mathbb{R}^4$, $f(t) = (\cos t, \sin t, 0, 0)$, obtained from $f(t) = (\cos t, \sin t)$, has its Frenet apparatus with curvatures $\kappa_1 = 1$, $\kappa_2 = 0$, $\kappa_3 = 0$ and the Frenet frame

$$\mathcal{F} = \{V_1, V_2, V_3, V_4\},$$

where $V_1 = (-\sin t, \cos t, 0, 0)$, $V_2 = (-\cos t, -\sin t, 0, 0)$, $V_3 = (0, 0, 1, 0)$ e $V_4 = (0, 0, 0, 1)$. Finally, note that Theorem 2.2 may be applied to any regular curve that has a well defined Frenet apparatus.

At the moment that we have a new theorem, it is always good to seek an example to highlight its statement. We do not find in the literature any non-trivial example of a genuine tube, even of disks, other than the torus of revolution. For this reason, we will try to expose such an example including a significant level of details. Much of the calculations will be omitted, but they will be indicated and left as exercises for the reader. We will construct a maximal genuine disk tube around the circular helix $f(t) = (\cos t, \sin t, t)$, $t \in \mathbb{R}$. The example will be decomposed in steps, some of them including some interesting lemmas of real analysis. Actually, we have here a surprisingly laborious and elegant example.

Example 2.4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^3$, $f(t) = (\cos t, \sin t, t)$. The apparatus of f is

[1] $\kappa = \kappa_1 = \frac{1}{2}$; **[2]** $\tau = \kappa_2 = \frac{1}{2}$;

[3] $\mathbf{T} = V_1 = \left(-\frac{\sin(t)}{\sqrt{2}}, \frac{\cos(t)}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$;

[4] $\mathbf{N} = V_2 = (-\cos(t), -\sin(t), 0)$;

[5] $\mathbf{B} = V_3 = \left(\frac{\sin(t)}{\sqrt{2}}, -\frac{\cos(t)}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

We will study the disk tube of radius 2 around f and centered at $P = (0, 0, t)$, which will be indicated by $\Gamma_1 = G^*(B^{[2]}[2] \times \mathbb{R})$, where $B^{[2]}[2] \subset \mathbb{R}^2$ is the compact disk of radius 2 centered at origin and G^* from $\mathbb{R}^3$ into $\mathbb{R}^3$ is

$$\begin{array}{rcl} G^*(X,t) & = & f(t) - x\,\mathbf{N}(t) + y\,\mathbf{B}(t) \\ & = & \left((x+1)\cos(t) + \frac{y\sin(t)}{\sqrt{2}}, (x+1)\sin(t) - \frac{y\cos(t)}{\sqrt{2}}, t + \frac{y}{\sqrt{2}}\right), \end{array}$$

where $X = (x, y)$. Looking at the map G in (2.1), we see that in the rule above we consider $-\mathbf{N}(t)$ instead $\mathbf{N}(t)$ (compare to the Remark 2.1). The reason is that this simplifies the handling of certain angles, mainly T_I and T (see the next picture). The disk tubes obtained from G^* and G coincide, since $G^*(-x, y, t)$ equals

$$G(x, y, t) = f(t) + x\mathbf{N}(t) + y\mathbf{B}(t),$$

which is the map in (2.1). Also, it is easy to see that the injectivity of one of them implies the injectivity of the other one. In what follows, we will conclude that G^* (and thus G) is injective in $B^{[2]}[2] \times \mathbb{R}$ and that this fails in $B^{[2]}(R) \times \mathbb{R}$ for any open disk of radius $R > 2$, $B^{[2]}(R)$. So, suppose that $G^*(A, t_1) = G^*(B, t_2)$, where $A = (x_1, y_1)$ and $B = (x_2, y_2)$ lie in $B^{[2]}[2]$ and, without loss of generality, $t_1 \leq t_2$. We are going to our first claim.

Claim 1:

[1] $t_2 - t_1 = \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}}$;

[2] $(x_1+1)^2 + \frac{y_1^2}{2} = (x_2+1)^2 + \frac{y_2^2}{2}$.

In fact, $G^*(x_1, y_1, t_1) = G^*(x_2, y_2, t_2)$ is the same as

$$\begin{pmatrix} (x_1+1)\cos(t_1) + \frac{y_1\sin(t_1)}{\sqrt{2}} \\ (x_1+1)\sin(t_1) - \frac{y_1\cos(t_1)}{\sqrt{2}} \\ t_1 + \frac{y_1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} (x_2+1)\cos(t_2) + \frac{y_2\sin(t_2)}{\sqrt{2}} \\ (x_2+1)\sin(t_2) - \frac{y_2\cos(t_2)}{\sqrt{2}} \\ t_2 + \frac{y_2}{\sqrt{2}} \end{pmatrix}$$

whence we get easily [1]. By summing the squares of the two first entries in the matrices above, we get [2], which shows that A and B lie in the ellipse $\mathcal{E}_r$

$$\mathcal{E}_r = \{(x,y) \in \mathbb{R}^2;\ H(x,y) = (x+1)^2 + \frac{y^2}{2} = r^2\},$$

where r^2 equals the identical real numbers in [2]. The ellipse $\mathcal{E}_r$ plays a crucial role in the discussion that follows. Since $0 = H(-1,0) \le H(x,y) \le 9 = H(2,0)$, for all $(x,y) \in B^{[2]}[2]$, and $(2,0)$ is the only maximum point of H on $B^{[2]}[2]$, we see that $0 \le r \le 3$ and, moreover, $r = 0$ implies that $A = B = (-1,0)$ and $r = 3$ yields $A = B = (2,0)$. In any case, [1] gives $t_1 = t_2$ and thus $(x_1, y_1, t_1) = (x_2, y_2, t_2)$. This is what we want. Hence, we may work with $0 < r < 3$. Some observations on the ellipse $\mathcal{E}_r$: its semi-minor axis is r and its semi-major axis is $\sqrt{2}\,r$; it is contained in the open disk $B^{[2]}(2)$, when $r < 1$; when $r = 1$, it is contained in the compact disk $B^{[2]}[2]$ and $\mathcal{E}_r \cap B^{[2]}[2] = \{(-2,0)\}$; if $r > 1$, the intersection $\mathcal{E}_r \cap B^{[2]}[2]$ equals the arc of the $\mathcal{E}_r$ joining the points I_1 and I_2, as in the picture below (the angle θ that appears in it will be considered later). Now, we parametrize $\mathcal{E}_r$ by

$$E(t) = (r\cos(t) - 1, r\sqrt{2}\sin(t)), \quad t \in [-\pi, \pi].$$

For the moment, we consider $1 < r < 3$. A direct calculation shows that

$$I_1 = (x_I, y_I) = \left(-2 + \sqrt{2}\sqrt{r^2-1}, \sqrt{2}\sqrt{1 - r^2 + 2\sqrt{2}\sqrt{r^2-1}}\right),$$

and $I_2 = (x_I, -y_I)$. We have unique angles $T_I \in (0,\pi)$ and $T \in (-\pi,\pi)$ such that $E(T_I) = I_1$ and $E(T) = A$ (there exists an analogous angle for B as well). Hence,

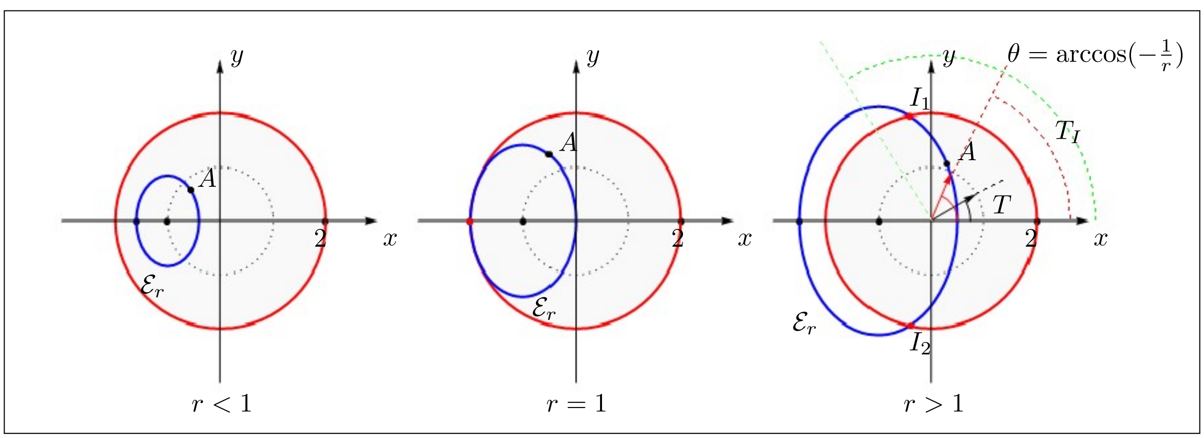

Figure 7. Geometric support for the Example 2.4

$$\begin{cases} \cos(T_I) = \dfrac{1+x_I}{r} = \dfrac{\sqrt{2}\sqrt{r^2-1}-1}{r}, \\ \sin(T_I) = \dfrac{y_I}{r\sqrt{2}} = \dfrac{\sqrt{1-r^2+2\sqrt{2}\sqrt{r^2-1}}}{r}, \\ \cos(T) = \dfrac{x_1+1}{r}, \\ \sin(T) = \dfrac{y_1}{r\sqrt{2}}. \end{cases} \tag{2.2}$$

Claim 2:

$$-T_I = -\arccos\left(\frac{\sqrt{2}\sqrt{r^2-1}-1}{r}\right) \leq T \leq T_I = \arccos\left(\frac{\sqrt{2}\sqrt{r^2-1}-1}{r}\right).$$

This claim follows from the discussion above.

Claim 3:

$$-T_I = -\arccos\left(\frac{\sqrt{2}\sqrt{r^2-1}-1}{r}\right) \leq T \leq T_I = \arccos\left(\frac{\sqrt{2}\sqrt{r^2-1}-1}{r}\right).$$

This claim follows from the discussion above.

Claim 4: without restrictions on r, we have that

$$x_1 \sin(t_2-t_1) + \frac{y_1(\cos(t_2-t_1)-1)}{\sqrt{2}} + t_2 - t_1 + \sin(t_2-t_1) = 0. \tag{2.3}$$

In fact, firstly

$$(f(t_2) - G^*(B,t_2)) \cdot f'(t_2) = (-x_2\,\mathbf{N}(t_2) + y_2\,\mathbf{B}(t_2)) \cdot f'(t_2) = 0.$$

On the other hand, since $G^*(B,t_2) = G^*(A,t_1)$,

$$\begin{aligned} 0 = (f(t_2) - G^*(B,t_2)) \cdot f'(t_2) &= (f(t_2) - G^*(A,t_1)) \cdot f'(t_2) \\ &= x_1 \sin(t_2-t_1) + \frac{y_1(\cos(t_2-t_1)-1)}{\sqrt{2}} + \\ &\quad + t_2 - t_1 + \sin(t_2-t_1), \end{aligned}$$

after some simplifications. Now, we get the fundamental function g of our example, obtained from the left side of (2.3) by putting $s = t_2 - t_1 \geq 0$ and by using the properties of the angle T in (2.2), namely:

$$g(s) = s - r\sin(T) + r\sin(s+T),\ s \geq 0.$$

Note that $g(0) = 0$. The main idea is to show that $s = 0$ is the only zero of g. Suppose, for a moment, that this is indeed true. The equation above, together with (2.3), says that $t_2 - t_1$ is a zero of g. Hence $t_1 = t_2$. By using [1] of the claim 1 and the equation (3), we deduce that

$$\begin{pmatrix} (x_1+1)\cos(t_1) \\ (x_1+1)\sin(t_1) \end{pmatrix} = \begin{pmatrix} (x_2+1)\cos(t_1) \\ (x_2+1)\sin(t_1) \end{pmatrix},$$

or

$$((x_1 - x_2)\cos(t_1), (x_1 - x_2)\sin(t_1)) = (0, 0).$$

Therefore, $x_1 - x_2 = 0$ and $(x_1, y_1, t_1) = (x_2, y_2, t_2)$, as we desire. In general, g can have positive zeros, but with the constraints that come from our problem, this cannot happen and, in fact, g must have only a zero, as we will see below.

In the cases $0 < r \leq 1$, the problem is very simple. Indeed, since

$$g'(s) = r\cos(s+T) + 1 \geq 1 - r \geq 0.$$

we get that g is increasing and its critical points are isolated, which implies that g is a strictly increasing function. In particular, $g(s) > 0$, for all $s > 0$. Hence, g has no positive zero and we are done: $(x_1, y_1, t_1) = (x_2, y_2, t_2)$. We are left with the cases $1 < r < 3$, which involves many technicalities. Firstly, since A and B belong to the disk $B^{[2]}[2]$, we have that the absolute values of their coordinates are lower than or equal to 2. Hence, by using [1] in the claim 1, we get $s = |s| = |t_2 - t_1| \leq 2\sqrt{2} < \pi$. Therefore, we can consider g defined in $[0, \pi]$:

$$g(s) = s - r\sin(T) + r\sin(s+T),\ 0 \leq s \leq \pi,$$

subject to the constraints

$$-\pi < -T_I \leq T \leq T_I < \pi,$$

where $T_I = \arccos(h(r))$,

$$h(r) = \frac{\sqrt{2}\sqrt{r^2-1} - 1}{r},$$

and always keeping in mind that

$$\cos(T) = \frac{x_1+1}{r}, \quad \sin(T) = \frac{y_1}{r\sqrt{2}},$$

$A = (x_1, y_1)$ and, thus, r and T are all fixed. Another important date in our discussion is the angle $\theta \in (\frac{\pi}{2}, \pi)$ (as shown in the Figure 7) given by $\theta = \arccos(-\frac{1}{r})$. Of course $\theta > T_I$, because $h(r) + \frac{1}{r} > 0$ and the arccos is a strictly decreasing function. Our strategy will be to describe the critical points of g and then to show that g is positive at each of these points, which implies that $g > 0$ on $(0, \pi]$, according to the next lemma (Lemma 2.5). We are going to describe the critical points of g. For this purpose, let $s_c \in (0, \pi)$ be a critical point of g.

Claim 5: the critical point s_c satisfies only one of the following descriptions:

[1] $s_c + T = \theta$;

[2] $s_c + T = -\theta$;

[3] $s_c + T = -\theta + 2\pi$.

Indeed, since $g'(s_c) = 1 + r\cos(s_c + T) = 0$, we get that either $s_c + T = \theta + 2k\pi$ or $s_c + T = -\theta + 2k\pi$, for some integer k. This, together with $\theta \in (\pi/2, \pi)$ and $s_c + T \in (-\pi, 2\pi)$, due to the constraints on s_c and T, implies the three items above.

Claim 6: $g(s_c) > 0$.

We will use separately each case in the claim 4.

[1] Here, $s_c + T = \theta$, whence

$$g(s_c) = s_c - r\sin(T) + r\sin(\theta) = (\theta + r\sin(\theta)) - (r\sin(T) + T) = \psi(\theta) - \psi(T),$$

where,

$$\psi(u) = u + r\sin(u).$$

Of course $g(s_c) > 0$, if $T \leq 0$. When

$$0 < T < T_I < \theta < \pi,$$

we need a more delicate analysis, as follows. It is not hard to see that ψ, on the interval $[0, \pi]$, attains its strict global maximum at $u = \theta$, its only critical point. Hence, $g(s_c) = \psi(\theta) - \psi(T) > 0$.

[2] Now, $s_c + T = -\theta$. Hence, $-\pi < s_c + T < -\frac{\pi}{2}$ and $\cos(s_c + T) = -1/r$. Suppose, by contradiction, that $g(s_c) \leq 0$. Since $g(0) = 0$ and

$$g'(0) = 1 + r\cos(|T|) \geq 1 + r\cos(T_I) = \sqrt{2}\sqrt{r^2 - 1} > 0,$$

we would have $g(s_1) = 0$, for some $0 < s_1 \leq s_c$. Therefore, we would have $g'(s_2) = 0$, for some $0 < s_2 < s_c$, hence $\cos(s_c + T) = \cos(s_2 + T) = -1/r$ and $-\pi < s_2 + T < s_c + T < -\pi/2$. We have an absurdity, because the cos is injective in $(-\pi, -\pi/2)$.

[3] $s_c + T = -\theta + 2\pi$. Actually, this is the harder part of our example. First, note that $T > 0$. We have

$$g\left(s_c\right) = g(-\theta - T + 2\pi) = -\theta - T + 2\pi - r\sin(T) - r\sin(\theta) = 2\pi - (\psi(\theta) + \psi(T)).$$

Now, since ψ (see (6)) is strictly increasing in $(0, \theta)$, we get $-\psi(T) > -\psi(T_I)$ and thus

$$g\left(s_c\right) = 2\pi - (\psi(\theta) + \psi(T)) > 2\pi - (\psi(\theta) + \psi(T_I)) = 2\pi - \eta(r),$$

where

$$\eta(u) = \left(\arccos\left(-\frac{1}{u}\right) + \sqrt{u^2 - 1}\right) + \\ + \left(\arccos\left(\frac{\sqrt{2}\sqrt{u^2-1} - 1}{u}\right) + \sqrt{1 - u^2 + 2\sqrt{2}\sqrt{u^2 - 1}}\right),$$

$u \in [1, 3]$, which we can patiently verify the following:

(i) $\eta(1) = 2\pi$;

(ii) $\eta(3) = 2\sqrt{2} + \arccos\left(-\frac{1}{3}\right) < 2\pi$;

(iii) $\eta'(u) = 0$ if and only if $u = \sqrt{3}$;

(iv)
$$\eta(\sqrt{3}) = 2\sqrt{2} + \arccos\left(-\frac{1}{\sqrt{3}}\right) + \arccos\left(\frac{1}{\sqrt{3}}\right) = 2\sqrt{2} + \pi < 2\pi,$$

for $\arccos(-x) + \arccos(x)$ is constant and equal to π.

These properties imply that $u = 1$ is the global maximum point for η. Hence, $\eta(u) < 2\pi$ in the interval $(1, 3)$. As a matter of fact, since $\eta(1) > \eta(\sqrt{3}) > \eta(3)$, η is strictly decreasing in $[1, 3]$. So $g(s_c) = 2\pi - \eta(u) > 0$.

We are almost done, because we still need to verify that $g(\pi) = \pi - 2r\sin(T) > 0$. The lemma that we will use requires this fact, which will be left as an exercise. We suggest to consider separately $1 < r < \sqrt{\frac{3}{2}}$, case in which $T_I > \frac{\pi}{2}$, and $\sqrt{\frac{3}{2}} \leq r < 3$, where $T_I \leq \frac{\pi}{2}$. Of course the result is trivial if $T \leq 0$. Putting this all together, we get that $g(0) = 0$, $g > 0$ at its critical points and $g(\pi) > 0$. By using lemma below, it follows that $g > 0$ in $(0, \pi)$. Thus, g vanishes only at $s = 0$ and it follows that $(X, t_1) = (Y, t_2)$ (see the main idea that we had talked above), that is, G^* and G are both injective.

It remains to show that Γ^* is a maximal genuine tube, that is, if $R > 2$ then G^* is no longer injective. In fact, let $R > 2$ and define $r = \frac{R}{2} > 1$. Then let s_0 be a positive solution of $s - r \sin s = 0$ (the left side is the function g for $T = \pi$) that exists, according to Lemma 2.6 bellow. Now put $P_1 = (X_1, 0)$, $X_1 = (0, -1 - r)$, and $P_2 = (Y_1, s_0)$, $Y_1 = (-1 - r\cos(s_0), \sqrt{2}\, s_0)$. Of course $P_1 \neq P_2$, $G^*(P_1) = (-r, 0, 0)$ and

$$G^*(P_2) = \left(r\frac{s_0^2}{r^2} - r - s_0\frac{s_0}{r}, s_0\cos(s_0) - r\frac{s_0}{r}\cos(s_0), 0 \right) = (-r, 0, 0) = G^*(P_1).$$

Moreover $\|X_1\| = 1 + r < R$, by the construction of r, and $\|Y_1\| < R$, because Y_1 belongs to the ellipse

$$(x+1)^2 + \frac{y^2}{2} = r^2 = \frac{R^2}{4},$$

whose intersection with the circle $x^2 + y^2 = R^2$ is empty, given that the system formed by these curves has no solution in $\mathbb{R}^2$. By working a little harder, we can establish that

$$G^*(0, -1 - r, t) = G^*(-1 - r\cos(s_0), \sqrt{2}\, s_0, t + s_0) = (-r\cos(t), -r\sin(t), t),$$

for an arbitrary t. Finally, we see below a piece of the infinite disk tube Γ^*, say Γ_1,

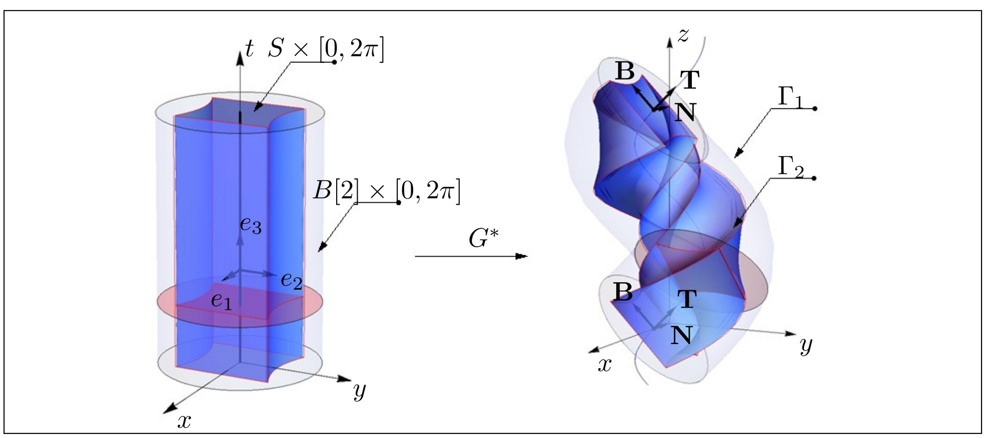

Figure 8. Genuine tubes around the circular helix

together with another genuine tube $\Gamma_2 \subset \Gamma_1$, whose cross sections arecopies of the region

$$S = \left\{(x, y);\ -1 \leq x \leq 1,\ -\frac{x^2}{4} - 1 \leq y \leq \frac{x^2}{4} + 1\right\}. \tag{2.4}$$

The tubes Γ_1 and Γ_2 are obtained for t running from 0 to 2π. In Example 3.4, we calculate their volumes.

Lemma 2.5. *Let $g : [0, b] \longrightarrow \mathbb{R}$ be a differentiable function such that $g(0) \geq 0$ and $g(b) > 0$. If either g has no critical point in $(0, b)$ or g is positive at each of its critical points, then $g > 0$ in $(0, b]$.*

Proof. Suppose, by contradiction, that $g(x_1) \leq 0$ for some $x_1 \in (0, b)$. Hence, there exists $x_0 \in [x_1, b)$ such the $g(x_0) = 0$, since $g(b) > 0$. By using the Rolle's theorem, if $g(0) = 0$, we get $x_c \in (0, x_0)$ such that $g'(x_c) = 0$. If $g(0) > 0$, the minimum value of g in $[0, b]$ is smaller than or equal to zero, because $g(x_0) = 0$ and thus this minimum value must occur at some $x_c \in (0, b)$, where g' vanishes. Hence, g must be positive in $(0, b]$, if g has no critical point. Now, we suppose that $g(x_c) > 0$, whenever $g'(x_c) = 0$. We have that $g'(x_0) \neq 0$, by hypothesis. If $g'(x_0) < 0$, we choose $x_1 > x_0$ where g is negative. This implies that the minimum value of g in $[x_0, b]$ is negative and occurs in (x_0, b), say at x_c. Hence $g'(x_c) = 0$ and $g(x_c) < 0$, a contradiction. The case $g'(x_0) > 0$ leads to a similar contradiction, now by taking $x_1 < x_0$ where g is negative. $\square$

Lemma 2.6. *If $r > 1$, then there exists $s_0 \in (0, \pi)$ such that $s_0 - r \sin s_0 = 0$.*

Proof. Define $g(s) = s - r \sin s$, $s \in \mathbb{R}$. If $s \neq 0$, $g(s) = s\left(1 - r\frac{\sin s}{s}\right)$ and, since

$$\lim_{s \to 0} \left(1 - r\frac{\sin s}{s}\right) = 1 - r < 0,$$

we can choose $s_1 > 0$ near to zero such that $g(s_1) < 0$. But $g(\pi) = \pi > 0$. From the Intermediate Value Theorem, we obtain s_0 between s_1 and π such that $g(s_0) = 0$, as we wanted. $\square$

3 On the Volume of Tubes

The calculation of the volume of disk tubes in $\mathbb{R}^{n+1}$ and in the euclidean sphere $\mathbb{S}^n$ appears in [8]. Actually, the main goal of such a paper is to work within the sphere. In [5], the author generalizes the Pappus theorems for genuine tubes around an arbitrary curve in $\mathbb{R}^3$. In the case of the second Pappus theorem, his proof makes use of the divergence theorem. Here, we will establish such a theorem for genuine tubes in $\mathbb{R}^{n+1}$ by using a different strategy, namely, the formula for change variable in multiple integrals.

Next, we will consider an interval $I = [a, b]$ and the tube $\Gamma = \Gamma_P(f, \Omega_I(S))$ generated by the solid $\Omega_I(S_t)$ around the curve $f : J \supset I \longrightarrow \mathbb{R}^{n+1}$ and centered at $P(t) = (\widetilde{P}, t)$, where $S_t \subset \mathbb{R}^n \times \{t\}$ has positive volume and $\widetilde{P} = (p_1, p_2, \ldots, p_n)$ is constant. Thus, the cross sections of Γ are stuck to f by the same point. We have that Γ is the image of $\Omega_I(S_t)$ under $G : \mathbb{R}^n \times J \longrightarrow \mathbb{R}^{n+1}$ given by

$$G(X, t) = f(t) + \sum_{j=1}^{n} (x_j - p_j)\, V_{j+1}(t), \tag{3.1}$$

where $(X, t) = (x_1, x_2, \ldots, x_n, t)$. Moreover, exactly as in the proof of Theorem 2.2, we have that

$$|\det JG(x_1, x_2, \ldots, x_n, t)| = \nu(t)|1 - (x_1 - p_1)\,\kappa_1(t)|.$$

Let $R \in \mathbb{R}$ be a positive number smaller than the *radius of curvature* $\rho = \frac{1}{\sup_J \kappa_1(t)}$. Let $B^{[n]}(\widetilde{P}, R)$ be the open disk centered at $\widetilde{P}$ of radius R in $\mathbb{R}^n$. Then, for all $X = (x_1, x_2, \ldots, x_n) \in B^{[n]}(\widetilde{P}, R)$ and $t \in J$, we have that

$$|\det JG(x_1, x_2, \ldots, x_n, t)| = \nu(t)|1-(x_1-p_1)\,\kappa_1(t)| = \nu(t)(1-(x_1-p_1)\,\kappa_1(t)) > 0, \tag{3.2}$$

because

$$(x_1 - p_1)\,\kappa_1 \le |x_1 - p_1| \sup_J \kappa_1 \le \left\| X - \widetilde{P} \right\| \sup_J \kappa_1 < \rho \sup_J \kappa_1 = 1.$$

We refer to the inequality in (3.2) as *fundamental regularity condition*. We refer to the inequality in (3.2) as *fundamental regularity condition*.

Remark 3.1. The formula above is obtained from the default map G in (2.1). For example, if we use the vector field $-V_2(t)$ instead $V_2(t)$ and consider the map

$$G^*(X, t) = f(t) - (x_1 - p_1(t))\, V_2(t) + \sum_{j=2}^{n} (x_j - p_j(t))\, V_{j+1}(t),$$

the absolute value of its jacobian becomes

$$|\det JG^*(x_1, x_2, \ldots, x_n, t)| = \nu(t)|1 + (x_1 - p_1)\,\kappa_1(t)|.$$

Now, suppose that $\Omega_I(S_t) \subset B^{[n]}(\widetilde{P}, R) \times J$, and that G is injective in $B^{[n]}(\widetilde{P}, R) \times J$. By using the change of variables formula for multiple integrals and the Fubini theorem, it follows that

$$\begin{aligned}
\operatorname{vol}\Gamma &= \int_\Gamma \mathrm{d}y_1\mathrm{d}y_2\ldots\mathrm{d}y_{n+1} = \int_{G(\Omega_I(S_t))} \mathrm{d}y_1\mathrm{d}y_2\ldots\mathrm{d}y_{n+1} \\
&= \int_{\Omega_I(S_t)} |\det JG|\,\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\,\mathrm{d}t \\
&= \int_a^b \nu(t)\left(\int_{S_t} (1-(x_1-p_1)\,\kappa_1(t))\,\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\right)\,\mathrm{d}t \\
&= \int_a^b \nu(t)(\operatorname{vol}S_t)\,\mathrm{d}t - \int_a^b \nu(t)\,\kappa_1(t)\left(\int_{S_t} x_1\,\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\right)\,\mathrm{d}t+ \\
&\qquad\qquad + p_1\int_a^b \nu(t)\,\kappa_1(t)(\operatorname{vol}S_t)\,\mathrm{d}t,
\end{aligned} \tag{3.3}$$

From this, we will get a series of interesting results on the volume of the tubes. For instance, under a reasonable condition on the injectivity of G, $\operatorname{vol}\Gamma$ does not change if we replace the $(n-1)$ last coordinates of $\widetilde{P}$ by any $(n-1)$-tuple. This means that we can shift the axis of the tube orthogonally to V_2 without altering the volume of the tube.

Now, we give a definition. Given a compact set $S \subset \mathbb{R}^n$ with $\operatorname{vol}S > 0$, we define the *barycenter of* S to be the point $\mathcal{B}(S) = (c_1, c_2, \ldots, c_n)$, where

$$\begin{aligned}
c_j &= \frac{1}{\int_S \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n}\int_S x_j\,\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n \\
&= \frac{1}{\operatorname{vol}S}\int_S x_j\,\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n,
\end{aligned} \tag{3.4}$$

$1 \le j \le n$. In some situations we use $\mathcal{B}(S, j)$ to mean c_j.

By using this definition, (3.3) becomes

$$\operatorname{vol}\Gamma = \int_a^b \nu(t)\operatorname{vol}S_t\,\mathrm{d}t - \int_a^b \nu(t)\,\kappa_1(t)\,(\operatorname{vol}S_t)\,\mathcal{B}(S_t,1)\,\mathrm{d}t + p_1\int_a^b \nu(t)\,\kappa_1(t)\operatorname{vol}S_t\,\mathrm{d}t.$$

In particular, if $S_t = S$ is constant,

$$\operatorname{vol}\Gamma = (\operatorname{vol}S)\,l(f,I) + (\operatorname{vol}S)\,(p_1 - \mathcal{B}(S,1))\int_a^b \nu(t)\,\kappa_1(t)\,\mathrm{d}t, \tag{3.5}$$

where $l(f, I)$ is the length of arc of the curve f from $t = a$ to $t = b$.

Proposition 3.2. *Given $A = (a_1, a_2, \ldots, a_n)$ and $R > 0$, the barycenter of the compact disk centered at A and of radius R, $B^{[n]}[A, R] \subset \mathbb{R}^n$, is its center.*

Proof. We can suppose without loss of generality that A is the origin of $\mathbb{R}^n$. Let $S = B^{[n]}[R]$. From

$$\int_S x_n \; \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n = \int_{B^{n-1}[R]} \left(\int_{-\sqrt{R^2 - \sum_{i=1}^{n-1} x_i^2}}^{\sqrt{R^2 - \sum_{i=1}^{n-1} x_i^2}} x_n \mathrm{d}x_n \right) \; \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1} = 0,$$

we get that $\mathcal{B}(S, n) = 0$. Now, for $1 \leq j \leq n-1$,

$$\begin{aligned} \int_S x_j \; \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n &= \int_{-R}^{R} \left(\int_{B^{n-1}[\sqrt{R^2 - x_n^2}]} x_j \; \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1} \right) \mathrm{d}x_n \\ &= \int_{-R}^{R} \Big(\mathrm{vol}(B^{n-1}[\sqrt{R^2 - x_n^2}]) \, \mathcal{B}(B^{n-1}[\sqrt{R^2 - x_n^2}], j) \Big) \, \mathrm{d}x_n \\ &= \int_{-R}^{R} 0 \mathrm{d}x_n = 0, \end{aligned}$$

by using induction on n. Hence, $\mathcal{B}(S)$ equals the origin of $\mathbb{R}^n$. □

The second Pappus theorem, as the original version as its generalization in [5], deals with the calculation of volumes of tubes with constant cross sections and axes passing trough the barycenters of these sections. In our case this means that $S_t = S$, $\Omega_I(S_t) = S \times I$, $\widetilde{P} = \mathcal{B}(S)$ and $\Gamma = G(S \times I)$, as in the picture bellow, where we are taking account all discussion just before equation (3.3). For such a genuine tube, we have the following theorem that generalizes the second Pappus theorem for tubes in $\mathbb{R}^{n+1}$.

Theorem 3.3. $\mathrm{vol}\, \Gamma = l(f, I) \, \mathrm{vol}\, S$, *where $l(f, I)$ is the length of the curve f from $t = a$ to $t = b$. In other words, the volume of the tube is the product of the length of the curve traveled by the barycenter of S and the volume of the cross section S.*

Proof. Initially, note that $p_1 = \mathcal{B}(S, 1)$. By replacing this fact in (3.5), we obtain easily that $\mathrm{vol}\, \Gamma = (\mathrm{vol}\, S) \, l(f, I)$, as we wanted. □

Another application of equation (3.5) is the calculation of volumes of the traditional and genuine disk tubes with variable radius, say $R(t)$, that is, for each $t \in J$, the cross section S_t equals the disk $B^{[n]}[R(t)]$ and $P(t) = (0, \ldots, 0, t)$ or $\widetilde{P}$ is the center of the disk $B^{[n]}[R(t)]$. In this situation, we have $p_1 = 0$ and $\mathcal{B}(S_t, 1) = 0$ which implies that

$$\operatorname{vol}\Gamma = \int_a^b \nu(t) \operatorname{vol}(B^{[n]}[R(t)])\, \mathrm{d}t = \operatorname{vol}(B^{[n]}) \int_a^b \nu(t)(R(t))^n\, \mathrm{d}t, \tag{3.6}$$

where $\operatorname{vol}(B^{[n]})$ is the volume of the compact unit disk which can be recursively obtained from

$$\operatorname{vol}(B^{[n]}) = \operatorname{vol}(B^{[n-1]}) \int_{-1}^{1} (1 - x^2)^{\frac{n-1}{2}}\, \mathrm{d}x,$$

where $\operatorname{vol}(B^{[1]}) = \operatorname{vol}([-1, 1]) = 2$, as can be seen in [4] (page 74). In particular, we get the volume of the revolution solid Γ_α generated by the plane curve $\alpha(t) = (R(t), 0, 0, \ldots, 0, t) \in \mathbb{R}^{n+1}$, $t \in [a, b]$, around the y_{n+1}-axis, namely,

$$\operatorname{vol}\Gamma_\alpha = \operatorname{vol}(B^{[n]}) \int_a^b (R(t))^n\, \mathrm{d}t,$$

because, in this case, $f(t) = (0, 0, \ldots, 0, t)$ an thus $\nu = 1$. This result appears in [1] (equation (2.2)).

Example 3.4. Consider the tubes Γ_1 and Γ_2 in Example 2.4. Γ_1 is a disk tube of constant radius, namely $R(t) = 2$. The speed $\nu = \sqrt{2}$. Thus its volume is

$$\operatorname{vol}(\Gamma_1) = \operatorname{area}(B^{[2]}[2])\, l(f, [0, 2\pi]) = 4\pi \int_0^{2\pi} \sqrt{2}\, \mathrm{d}t = 8\sqrt{2}\pi^2.$$

For Γ_2, at first, note that $\mathcal{B}(S) = (0, 0)$ and that

$$\operatorname{area} S = 2 \int_{-1}^{1} \left(\frac{x^2}{4} + 1 \right) \mathrm{d}x = \frac{13}{3}.$$

Hence

$$\operatorname{vol}(\Gamma_2) = \operatorname{area}(S)\, l(f, [0, 2\pi]) = \frac{26\sqrt{2}\pi}{3}.$$

In both cases, we have used Theorem 3.3, because the axis of the tube (the circular helix) passes trough the barycenter of the cross section, that is, $\widetilde{P} = \mathcal{B}(S)$. In the examples that we have discussed so far, $\widetilde{P}$ has been chosen to be the barycenter of the cross section S. We now make a more instructive example. Indeed, consider the same cross section S as that of the tube Γ_2 (see (2.4)) and $\widetilde{P} = (\frac{1}{2}, 0)$. Now, it is exactly at this point that the circular helix $f(t) = (\cos t, \sin t, t)$, $t \in [0, 2\pi]$, will meet S to form the tube Γ_3. We have that $p_1 = \frac{1}{2}$ and $\mathcal{B}(S,1) = 0$. A full geometric description of this case is in the picture bellow, where $\Gamma_3 = G(S \times [0, 2\pi])$ and the default map G is given by

$$G(x, y, t) = f(t) + \left(x - \frac{1}{2}\right) \mathbf{N}(t) + y\,\mathbf{B}(t),$$

that is injective in $B^{[2]}[\widetilde{P}, 2] \times \mathbb{R}$ by the result in Example 2.4. Since $S \subset B^{[2]}[\widetilde{P}, 2]$, it follows that Γ_3 also is genuine and its volume is

$$\begin{aligned} \operatorname{vol}(\Gamma_3) &= (\operatorname{area} S)\, l(f, I) + (\operatorname{area} S)\,(p_1 - \mathcal{B}(S,1)) \int_a^b \nu(t)\,\kappa_1(t)\,\mathrm{d}t \\ &= \frac{13}{3}\, 2\pi\sqrt{2} + \frac{13}{3}\left(\frac{1}{2} - 0\right) 2\pi\,\sqrt{2}\,\frac{1}{2} = \frac{26}{3}\,\pi\sqrt{2} + \frac{13\pi\sqrt{2}}{6} = \frac{65\pi\sqrt{2}}{6}. \end{aligned}$$

Observe that the disk tube $G(B^{[2]}[\widetilde{P}, 2] \times [0, 2\pi])$ coincides with Γ_1, that was obtained with another map in Example 2.4. For completeness, let us include one more example, namely, a disk tube of variable radius around the same arc of the circular helix f. For this, consider $R(t) = 1 + \frac{1}{2}\sin t$ and Γ_4 the disk tube of radius $R(t)$, $0 \leq t \leq 2\pi$. This is to be to say that the cross sections of Γ_4 are the disks $S_t = B^{[2]}[R(t)]$. Since $0 < R(t) < 2$, $\Gamma_4 = G(\Omega_{[0.2\pi]} S_t)$ is genuine, where, of course, G must be changed to

$$G(x, y, t) = f(t) + x\,\mathbf{N}(t) + y\,\mathbf{B}(t).$$

By using (3.6), we get that

$$\operatorname{vol}(\Gamma_4) = \operatorname{area}(B^{[2]}) \int_a^b \nu(t)(R(t))^2\,\mathrm{d}t = \pi \int_0^{2\pi} \sqrt{2}\left(\frac{\sin(t)}{2} + 1\right)^2 \mathrm{d}t = \frac{9\pi^2}{2\sqrt{2}}.$$

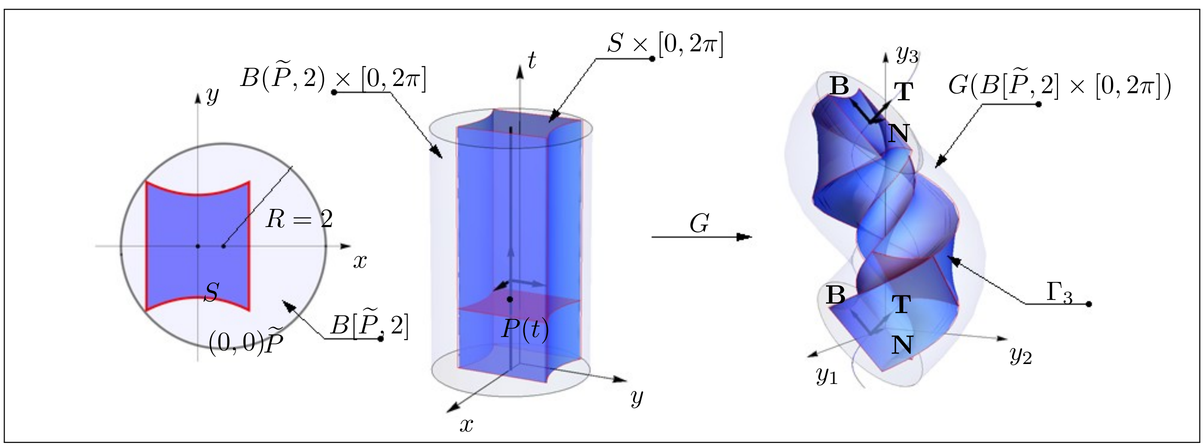

Figure 9. Tubes of Example 3.4

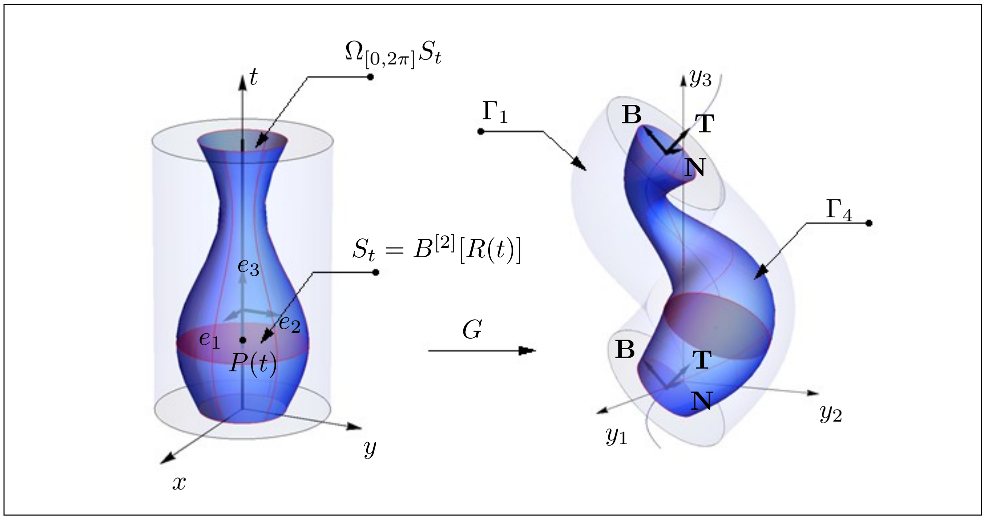

Figure 10. Disk Tubes of Example 3.4

4 On the Volume (Area) of Sphere Tubes

We start with a genuine compact disk tube $\Gamma \subset \mathbb{R}^{n+1}$ of variable radius, say $R(t) > 0$, and centered at the centers of its cross sections (see the left picture in Figure 10 above). So, we have a compact solid of revolution around the t-axis

$$\Omega_I = \Omega_I(B^{[n]}[R(t)]) = \{(X,t);\ \|X\| \leq R(t)\},$$

where $X = (x_1, x_2, \ldots, x_n)$ and $t \in I = [a,b]$, a curve f, with Frenet frame

$$\{\, V_1(t),\, V_2(t), \ldots,\, V_{n+1}(t)\},$$

defined in $J \supset I$, and the default map G,

$$G(X,t) = f(t) + \sum_{j=1}^{n} x_j\, V_{j+1}(t).$$

Next, we suppose that G is injective, satisfies the fundamental regularity condition (3.2) in some neighborhood of the Ω_I and that $\Gamma = G(\Omega_I)$. Denoting by $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$, we see that the boundary of Ω_I is given by

$$\partial\Omega_I = (B^{[n]}[R(a)], a) \cup \partial_l\Omega_I \cup (B^{[n]}[R(b)], b),$$

where

$$\partial_l \Omega_I = \{(R(t)\,X, t);\ X \in \mathbb{S}^{n-1}\},$$

$X = (x_1, x_2, \ldots, x_n)$ and $t \in I = [a, b]$, is what we call the *lateral hypersurface* of Ω_I. The image of this hypersurface of revolution under G will be called *lateral hypersurface* of Γ and will be denoted by $\partial_l\Gamma$. It is a *sphere tube*. Thus, if

$$\varphi(U) = (\varphi_1(U), \varphi_2(U), \ldots, \varphi_n(U)),$$

$U = (u_1, u_2, \ldots, u_{n-1})$, is a parametrization for $\mathbb{S}^{n-1}$, which we may take as an orthogonal parametrization, then

$$H(U, t) = G(\varphi(U), t) = f(t) + R(t) \sum_{j=1}^{n} \varphi_j(U)\, V_{j+1}(t), \tag{4.1}$$

U ranging over some region of $\mathbb{R}^{n-1}$, describes the hypersurface $\partial_l\Gamma$. Our main goal in this section is to get the volume (or area) element of $\partial_l\Gamma$ associated to H, that is,

$$\mathrm{d}(\partial_l\Gamma) = \sqrt{\det(g_{ij})}\ \mathrm{d}u_1\, \mathrm{d}u_2 \ldots \mathrm{d}u_{n-1}\, \mathrm{d}t,$$

where (g_{ij}) is the symmetric matrix of the first fundamental form of $\partial_l\Gamma$ with respect to H. More precisely, $g_{ij} = \frac{\partial H}{\partial u_i} \cdot \frac{\partial H}{\partial u_j}$, $1 \le i, j \le n-1$, $g_{in} = \frac{\partial H}{\partial u_i} \cdot \frac{\partial Y}{\partial t}$, $1 \le i \le n-1$, and $g_{nn} = \frac{\partial H}{\partial t} \cdot \frac{\partial H}{\partial t}$. We remember that

$$\sqrt{\det(g_{ij})} = \left\| \frac{\partial H}{\partial u_1} \wedge \frac{\partial H}{\partial u_2} \wedge \cdots \wedge \frac{\partial H}{\partial u_{n-1}} \wedge \frac{\partial H}{\partial t} \right\| = \left\| \frac{\partial H}{\partial u_1} \times \frac{\partial H}{\partial u_2} \times \cdots \times \frac{\partial H}{\partial u_{n-1}} \times \frac{\partial H}{\partial t} \right\|$$

and that

$$N(U, t) = N(u_1, u_2, \ldots, u_{n-1}, t) = \frac{\frac{\partial H}{\partial u_1} \times \frac{\partial H}{\partial u_2} \times \cdots \times \frac{\partial H}{\partial u_{n-1}} \times \frac{\partial H}{\partial t}}{\sqrt{\det(g_{ij})}}$$

is the Gauss map of $\partial_l\Gamma$. The next Lemma will be useful for our objective.

Lemma 4.1. *Let D be the following $(n \times n)$ "almost" diagonal symmetric matrix*

$$D = \begin{pmatrix} a_1 & 0 & \ldots & 0 & b_1 \\ 0 & a_2 & \ldots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & a_{n-1} & b_{n-1} \\ b_1 & b_2 & \ldots & b_{n-1} & a_n \end{pmatrix},$$

where $a_j \neq 0$, for all j. Then

$$\det\, D = a_1 \cdots a_{n-1}\left(a_n - \left(\frac{b_1^2}{a_1} + \cdots + \frac{b_{n-1}^2}{a_{n-1}}\right)\right).$$

Proof. After some elementary operations on the rows of D, we obtain that $\det D$ equals

$$a_1 \cdots a_{n-1} \det \begin{pmatrix} 1 & 0 & \dots & 0 & \frac{b_1}{a_1} \\ 0 & 1 & 0 & \dots & \frac{b_2}{a_2} \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & \frac{b_{n-1}}{a_{n-1}} \\ 0 & 0 & \dots & 0 & a_n - \frac{b_1^2}{a_1} - \frac{b_2^2}{a_2} - \cdots - \frac{b_{n-1}^2}{a_{n-1}} \end{pmatrix},$$

whence the result follows easily. □

Now, let us return to the function H in (4.1), where we consider φ as an orthogonal parametrization of $\mathbb{S}^{n-1}$, that is, $\frac{\partial \varphi}{\partial u_i} \cdot \frac{\partial \varphi}{\partial u_j} = 0$, whenever $i \neq j$. In other words, given $U = (u_1, \dots, u_{n-1})$, the set

$$\left\{\frac{\partial \varphi}{\partial u_1}, \frac{\partial \varphi}{\partial u_2}, \dots, \frac{\partial \varphi}{\partial u_{n-1}}\right\}$$

is an orthogonal basis of the tangent space of $\mathbb{S}^{n-1}$ at $\varphi(U)$. Moreover, since $\varphi \cdot \varphi = 1$, it follows that $\varphi \cdot \frac{\partial \varphi}{\partial u_j} = 0$, $1 \leq j \leq n-1$. Now, we will study the properties of H.

Proposition 4.2. *The parametrization H of $\partial_l \Gamma$ has the following properties, where the H_0 stands for the vector $H(U,t) - f(t)$.*

(i) $\|H_0\|^2 = (R(t))^2$;

(ii) $\frac{\partial H}{\partial u_i} \cdot \frac{\partial H}{\partial u_j} = 0$, $i \neq j$, $1 \leq i, j \leq n-1$;

(iii) $H_0 \cdot \frac{\partial H}{\partial u_j} = 0$, $i \neq j$, $1 \leq i, j \leq n-1$;

(iv) $H_0 \cdot \frac{\partial H}{\partial t} = R(t)R'(t)$;

(v)
$$\operatorname{span}\left\{H_0, \frac{\partial H}{\partial u_1}, \dots, \frac{\partial H}{\partial u_{n-1}}\right\} = \operatorname{span}\{\, V_2(t),\, V_3(t), \dots,\, V_{n+1}(t)\};$$

(vi)

$$\frac{\partial H}{\partial t}(U,t) = \nu(t)\,(1 - R(t)\varphi_1(U)\,\kappa_1(t))\;V_1(t) + \widetilde{\frac{\partial H}{\partial t}}(U,t),$$

where

$$\widetilde{\frac{\partial H}{\partial t}}(U,t) \in \operatorname{span}\{\,V_2(t),\,V_3(t),\ldots,\,V_{n+1}(t)\}.$$

Proof. The properties (i), (ii) and (iii) follow easily from those of φ, which we cited above. Differentiating (i) with respect to t, we get $H_0 \cdot \frac{\partial H_0}{\partial t} = R(t)R'(t)$. But

$$\frac{\partial H_0}{\partial t} = \frac{\partial H}{\partial t} - \nu(t)\,V_1(t)$$

and H_0 is perpendicular to V_1. Hence (iv) holds true. For (v), observe that

$$\operatorname{span}\left\{H_0, \frac{\partial H}{\partial u_1}, \ldots, \frac{\partial H}{\partial u_{n-1}}\right\} \subset \operatorname{span}\{\,V_2(t),\ldots,\,V_{n+1}(t)\}$$

and that both subspaces have the same dimension. The alternative (vi) is obtained by a direct calculation, by using the Frenet equations. □

Now, we can establish the main result of this section, where, for simplicity, in the larger formulas, we will use $R = R(t)$, $\kappa_1 = \kappa_1(t)$, $V_1 = V_1(t)$, $\phi_1 = \phi_1(U)$ and $U = (u_1, u_2, \ldots, u_{n-1})$.

Theorem 4.3. (i) *The volume (area) element* $\mathrm{d}(\partial_l\Gamma)$ *is equal to*

$$R^{n-1}\,\mathrm{d}\mathbb{S}^{n-1}\sqrt{\nu^2\,(1-\varphi_1\,R\,\kappa_1)^{\,2} + (R')^2}\;\mathrm{d}t,$$

where $\mathrm{d}\mathbb{S}^{n-1}$ *is the volume (area) element of the unit sphere* $\mathbb{S}^{n-1}$*;*

(ii)

$$N(U,t) = \pm\frac{\nu\,(\varphi_1 R\,\kappa_1 - 1)\left(H_0 + \frac{RR'}{\nu(\varphi_1 R\,\kappa_1 - 1)}\,V_1\right)}{R\sqrt{\nu^2\,(\varphi_1 R\,\kappa_1 - 1)^2 + R'^2}}.$$

In particular, if $\partial_l\Gamma$ *is of revolution, then*

$$\mathrm{d}(\partial_l\Gamma) = (R(t))^{n-1}\,\mathrm{d}\mathbb{S}^{n-1}\sqrt{1 + (R'(t))^2}\;\mathrm{d}t,$$

where $\mathrm{d}\mathbb{S}^{n-1}$ *is the volume element of the unit sphere* $\mathbb{S}^{n-1}$*, and hence*

$$\mathrm{vol}(\partial_l\Gamma) = \mathrm{vol}(\mathbb{S}^{n-1}) \int_a^b (R(t))^{n-1}\sqrt{1+(R'(t))^2}\ \mathrm{d}t.$$

Also, in this case, the Gauss map is

$$N(U,t) = \pm\frac{1}{\sqrt{1+(R'(t))^2}}\left(\varphi(U), -R'(t)\right).$$

Proof. We start by setting $U_0 = H_0/R(t)$, $a_j = \left\|\frac{\partial H}{\partial u_j}\right\|^2$, $U_j = \frac{\partial H}{\partial u_j}/\sqrt{a_j}$, $b_j = \frac{\partial H}{\partial u_j}\cdot\frac{\partial H}{\partial t}$, for j running from 1 to $n-1$, and $a_n = \left\|\frac{\partial H}{\partial t}\right\|^2$. The vectors U_j, $0 \le j \le n-1$, form an orthonormal basis for the subspace in (v) of Proposition 4.2. We have that (g_{ij}), the matrix of the first fundamental form of $\partial_l\Gamma$ with respect to H, is equal to the matrix D of Lemma 4.1. From this lemma it follows that

$$\begin{aligned}\det(g_{ij}) &= \left\|\frac{\partial H}{\partial u_1}\right\|^2\left\|\frac{\partial H}{\partial u_2}\right\|^2\cdots\left\|\frac{\partial H}{\partial u_{n-1}}\right\|^2\ \left(\left\|\frac{\partial H}{\partial t}\right\|^2 - \sum_{j=1}^{n-1}\left(\frac{\partial H}{\partial t}\cdot U_j\right)^2\right)\\ &= \left\|\frac{\partial H}{\partial u_1}\right\|^2\left\|\frac{\partial H}{\partial u_2}\right\|^2\cdots\left\|\frac{\partial H}{\partial u_{n-1}}\right\|^2\ \left(\left\|\frac{\partial H}{\partial t}\right\|^2 - \sum_{j=1}^{n-1}\left(\widetilde{\frac{\partial H}{\partial t}}\cdot U_j\right)^2\right)\end{aligned} \tag{4.2}$$

Note that,

$$\left\|\frac{\partial H}{\partial t}\right\|^2 - \sum_{j=1}^{n-1}\left(\widetilde{\frac{\partial H}{\partial t}}\cdot U_j\right)^2 = \nu(t)^2\left(1 - R(t)\varphi_1(U)\,\kappa_1(t)\right)^2 + \left\|\widetilde{\frac{\partial H}{\partial t}}\right\|^2 - \sum_{j=1}^{n-1}\left(\widetilde{\frac{\partial H}{\partial t}}\cdot U_j\right)^2,$$

by (vi) above, and

$$\left\|\widetilde{\frac{\partial H}{\partial t}}\right\|^2 = \left(\widetilde{\frac{\partial H}{\partial t}}\cdot U_0\right)^2 + \sum_{j=1}^{n-1}\left(\widetilde{\frac{\partial H}{\partial t}}\cdot U_j\right)^2,$$

because $\widetilde{\frac{\partial H}{\partial t}} \in \mathrm{span}\{U_0, U_1, \ldots, U_{n-1}\}$. Moreover,

$$\left(\widetilde{\frac{\partial H}{\partial t}}\cdot U_0\right)^2 = \left(\frac{\partial H}{\partial t}\cdot\frac{H_0}{R(t)}\right)^2 = (R'(t))^2,$$

where we use (iv) of Proposition 4.2. Putting this all together in (4.2), we obtain that

$$\begin{aligned}\det(g_{ij}) &= \left\|\frac{\partial H}{\partial u_1}\right\|^2 \left\|\frac{\partial H}{\partial u_2}\right\|^2 \cdots \left\|\frac{\partial H}{\partial u_{n-1}}\right\|^2 \left(\nu(t)^2\left(1 - R(t)\varphi_1(U)\,\kappa_1(t)\right)^2 + (R'(t))^2\right)\\ &= \left\|R\frac{\partial \varphi}{\partial u_1}\right\|^2 \left\|R\frac{\partial \varphi}{\partial u_2}\right\|^2 \cdots \left\|R\frac{\partial \varphi}{\partial u_{n-1}}\right\|^2 \left(\nu^2\left(1 - R\varphi_1(U)\,\kappa_1\right)^2 + (R')^2\right)\\ &= (R)^{2(n-1)} \left\|\frac{\partial \varphi}{\partial u_1}\right\|^2 \left\|\frac{\partial \varphi}{\partial u_2}\right\|^2 \cdots \left\|\frac{\partial \varphi}{\partial u_{n-1}}\right\|^2 \left(\nu^2\left(1 - R\varphi_1(U)\,\kappa_1\right)^2 + (R')^2\right),\end{aligned}$$

from which $\mathrm{d}(\partial_l\Gamma)$ can be easily seen. For the Gauss map, we start by remembering that the vectors H_0 and V_1 are perpendicular to $\frac{\partial H}{\partial u_j}$, for $1 \leq j \leq n-1$, and that $H_0 \cdot \frac{\partial H}{\partial t} = RR'$. Thus, it is natural try to find some $\lambda \in \mathbb{R}$ such that $H_0 + \lambda V_1$ also is perpendicular to $\frac{\partial H}{\partial t}$. By solving the equation $(H_0 + \lambda V_1)\cdot \frac{\partial H}{\partial t} = 0$, we get that

$$\lambda = \frac{R(t)\,R'(t)}{\nu(t)\,\left(\varphi_1(U)R(t)\,\kappa_1(t) - 1\right)}.$$

Hence N equals $\pm\frac{H_0+\lambda V_1}{\|H_0+\lambda V_1\|}$. For the case when the hypersurface $\partial_l\Gamma$ is of revolution, we just do $f(t) = (0, 0, \dots, t)$, $\nu = 1$, $V_1 = (0, 0, \dots, 1)$, $\kappa_1 = 0$ and $V_j = e_{j-1}$, for $2 \leq j \leq n+1$, in (i) and (ii), where e_j is the j-th element of the canonical basis of $\mathbb{R}^{n+1}$. To end this proof, we remark that the Gauss map N is well defined due to the regularity condition (3.2), which implies that

$$(1 - \varphi_1(U)R(t)\,\kappa_1(t)) > 0,$$

because $|\varphi_1(U)| \leq 1$. Thus $\partial_l\Gamma$ is, in fact, a piece of a regular hypersurface of $\mathbb{R}^{n+1}$. We are done. □

We have the following corollary which shows that the first theorem of Pappus holds true for sphere tubes of radius constant in $\mathbb{R}^{n+1}$.

Corollary 4.4. *If $R(t) = R$ is constant, then*

$$\mathrm{vol}(\partial_l\Gamma) = \mathrm{vol}(\mathbb{S}^{n-1}(R))\, l(f, I),$$

where $\mathbb{S}^{n-1}(R)$ is the sphere of radius R centered at origin.

Proof. From (i) of Theorem 4.3, we get that

$$\begin{aligned}\mathrm{vol}(\partial_l\Gamma) = \int_{\partial_l\Gamma} \mathrm{d}(\partial_l\Gamma) &= R^{n-1}\int_a^b \nu(t)\left(\int_{\mathbb{S}^{n-1}} \mathrm{d}\mathbb{S}^{n-1}\,(1-\varphi_1(U)R\,\kappa_1(t))\right)\mathrm{d}t\\ &= R^{n-1}\mathrm{vol}(\mathbb{S}^{n-1})\,l(f,I) - R^n\int_a^b \nu(t)\kappa_1(t)\left(\int_{\mathbb{S}^{n-1}}\varphi_1(U)\,\mathrm{d}\mathbb{S}^{n-1}\right)\mathrm{d}t\\ &= \mathrm{vol}(\mathbb{S}^{n-1}(R))\,l(f,I),\end{aligned}$$

because $\int_{\mathbb{S}^{n-1}}\varphi_1(U)\,\mathrm{d}\mathbb{S}^{n-1} = 0$, according to the next lemma.

Lemma 4.5. *$\int_{\mathbb{S}^{n-1}} x_j\,\mathrm{d}\mathbb{S}^{n-1} = 0$, for $1 \le j \le n$, that is, the barycenter of the sphere is its center.*

Proof. Consider the $(n-1)$-form

$$\omega = \sum_{i=1}^{n}(-1)^{n-1}x_i\;\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\cdots\wedge\widehat{\mathrm{d}x_i}\wedge\cdots\wedge\mathrm{d}x_n,$$

where the hat $\widehat{\ }$ over a 1-form means that the correspondent 1-form is excluded from the wedge product. As we know, the restriction of ω to $\mathbb{S}^{n-1}$ equals $\mathrm{d}\mathbb{S}^{n-1}$ and $\mathrm{d}\omega = n\,\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\ldots\wedge\mathrm{d}x_n$. Hence,

$$\begin{aligned}\mathrm{d}(x_j\,\mathrm{d}\mathbb{S}^{n-1}) = \mathrm{d}(x_j\,\omega) &= \mathrm{d}x_j\wedge\omega + n\,x_j\;\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\ldots\wedge\mathrm{d}x_n\\ &= (-1)^{j-1}\,\mathrm{d}x_j\wedge(\,\mathrm{d}x_1\wedge\cdots\wedge\widehat{\mathrm{d}x_j}\wedge\cdots\wedge\mathrm{d}x_n) + n\,x_j\;\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n\\ &= (n+1)\,x_j\;\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\ldots\wedge\mathrm{d}x_n.\end{aligned}$$

Now, the theorem of Stokes gives us that

$$\int_{\mathbb{S}^{n-1}} x_j\,\mathrm{d}\mathbb{S}^{n-1} = (n+1)\int_{B^{[n]}} x_j\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n = 0,$$

by using the Proposition 3.2. Here $B^{[n]}\subset\mathbb{R}^n$ is the compact unit disk. □

Remark 4.6. The arguments above show that $\mathrm{vol}(\mathbb{S}^{n-1}(R)) = n\,R^{n-1}\,\mathrm{vol}(B^{[n]})$.

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